ONE FACTOR GAUSSIAN SHORT RATE MODEL IMPLEMENTATION

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Abstract. We collect some results in Piterbarg, Interest Rate Modelling, needed for the implementation of a GSR model. We develop explicit formulas for piecewise constant volatility and reversion parameters under the forward measure.

1. model

The short rate dynamics is given by

$$dr(t) = \kappa(t)(\theta(t) - r(t))dt + \sigma_r(t)dW(t) \tag{1.1}$$

under the risk neutral measure. κ, σ_r are piecewise constant. Setting $x(t) := r(t) - f(t,t)$ with $f(t,T)$ denoting the instanteous forward rate observed at t for $T > t$, the dynamics can be rewritten

$$dx(t) = (y(t) - \kappa(t)x(t))dt + \sigma_r(t)dW(t) \tag{1.2}$$

with deterministic

$$y(t) = \int_0^t e^{-2\int_u^t \kappa(s)ds}\sigma_r(u)^2 du \tag{1.3}$$

In the $T-$forward measure the dynamics becomes

$$dx(t) = (y(t) - \sigma_r(t)^2 G(t,T) - \kappa(t)x(t))dt + \sigma_r(t)dW^T(t) \tag{1.4}$$

with

$$G(t,t') = \int_t^{t'} e^{-\int_t^u \kappa(s)ds} du \tag{1.5}$$

This fits into the general treatment under 2.1 with

$$a(t) = -\kappa(t) \tag{1.6}$$
$$b(t) = y(t) - \sigma_r(t)^2 G(t,T) \tag{1.7}$$
$$c(t) = \sigma_r(t) \tag{1.8}$$

Zero bond prices can be expressed as follows

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$$P(t,t') = \frac{P(0,t')}{P(0,t)} e^{-x(t)G(t,t') - \frac{1}{2}y(t)G(t,t')^2} \tag{1.9}$$

2. Basic SDE Integration

Consider the SDE

$$dx(t) = (a(t)x(t) + b(t))dt + c(t)dW(t) \tag{2.1}$$

with deterministic scalar functions a, b, c. The following is an explicit solution of 2.1.

$$x(t) = e^{\int_0^t a(u)du} \left(x(0) + \int_0^t e^{-\int_0^s a(u)du} b(s)ds + \int_0^t e^{-\int_0^s a(u)du} c(s)dW(s) \right) \tag{2.2}$$

That means that for $w < t$ $x(t)$ conditional on $x(w)$ is normally distributed with mean and variance given by

$$E(x(t)|x(w)) = A(w,t)x(w) + \int_w^t A(s,t)b(s)ds \tag{2.3}$$

$$\mathrm{Var}(x(t)|x(w)) = \int_w^t A(s,t)^2 c(s)^2 ds \tag{2.4}$$

with short hand notation

$$A(s,t) = e^{\int_s^t a(u)du} \tag{2.5}$$

3. Formulas for the piecewise constant case

Under the assumption of piecewise constant κ, σ_r we are left with the computation of some integrals for which we derive closed form formulas here. We fix a grid $0 = t_0 < t_1 < ... < t_n = T$ such that κ and σ_r are constant on each interval $[t_i, t_{i+1})$ and equal to $\kappa_i, \sigma_{r,i}$. We introduce the following notation: $l(t)$ denotes the largest index such that $t_{l(t)} \leq t$. Likewise $h(t)$ denotes the smallest index such that $t_{h(t)} \geq t$. Moreover we set $t_{i,s} := \max(t_i, s)$ and $t_{s,i} := \min(t_i, s)$.

3.1. Formula for $G(t,t')$.

We start with 1.5, the formula for $G(t,t')$. The integrand is

$$e^{-\int_t^u \kappa(s)ds} = \prod_{i=l(t)}^{h(u)-1} e^{-\kappa_i \int_{t_{i,t}}^{t_{u,i+1}} ds} = \prod_{i=l(t)}^{h(u)-1} e^{-\kappa_i (t_{u,i+1} - t_{i,t})} \tag{3.1}$$

By this

$$G(t,t') = \sum_{i=l(t)}^{h(t')-1} \int_{t_{i,t}}^{t_{t',i+1}} \left(\prod_{j=l(t)}^{i-1} e^{-\kappa_j (t_{j+1} - t_{j,t})} \right) e^{-\kappa_i (u - t_{i,t})} du \tag{3.2}$$

which is

$$\sum_{i=l(t)}^{h(t')-1} \left(\frac{1 - e^{-\kappa_i (t_{t',i+1} - t_{i,t})}}{\kappa_i} \right) \prod_{j=l(t)}^{i-1} e^{-\kappa_j (t_{j+1} - t_{j,t})} \tag{3.3}$$

We abbreviate this by

$$G(t,t') = \sum_{i=l(t)}^{h(t')-1} \eta_i \prod_{j=l(t)}^{i-1} \gamma_j \tag{3.4}$$

γ_j is dependent on t if and only if $j = l(t)$. In this case $i > l(t)$ necessarily. η_i is dependent on t if and only if $i = l(t)$. In these cases

$$\int \gamma_j dt = \frac{e^{-\kappa_j (t_{j+1} - t)}}{\kappa_j} \tag{3.5}$$

$$\int \eta_i dt = \frac{t\kappa_i - e^{-\kappa_i (t_{t',i+1} - t)}}{\kappa_i^2} \tag{3.6}$$

If $j > l(t)$ resp. $i > l(t)$ these integrals can be computed trivially by multiplying γ_j resp. η_i by the interval length over which the integral is computed.

3.2. **Formula for $y(t)$.** We continue with 1.3. The integrand here is

$$e^{-2\int_u^t \kappa(s)ds} \sigma_r(u)^2 = \prod_{i=l(u)}^{h(t)-1} \sigma_{r,i}^2 e^{-2\kappa_i (t_{t,i+1} - t_{i,u})} \tag{3.7}$$

We get

$$y(t) = \sum_{i=0}^{h(t)-1} \int_{t_i}^{t_{t,i+1}} \left(e^{-2\kappa_i (t_{t,i+1} - u)} \prod_{j=i+1}^{h(t)-1} \sigma_{r,j}^2 e^{-2\kappa_j (t_{t,j+1} - t_j)} \right) du \tag{3.8}$$

which is

$$\sum_{i=0}^{h(t)-1} \left(\frac{\sigma_{r,i}^2}{2\kappa_i} \left[1 - e^{-2\kappa_i (t_{t,i+1} - t_i)} \right] \prod_{j=i+1}^{h(t)-1} e^{-2\kappa_j (t_{t,j+1} - t_j)} \right) \tag{3.9}$$

and which we abbreviate by

$$y(t) = \sum_{i=0}^{h(t)-1} \left(\alpha_i \prod_{j=i+1}^{h(t)-1} \beta_j \right) \tag{3.10}$$

α_i resp. β_j is dependent on t if and only if $i = h(t) - 1$ resp. $j = h(t) - 1$. In the latter case $i < h(t) - 1$ necessarily. In these cases

$$\int \alpha_i dt = \frac{\sigma_{r,i}^2 (t\kappa_i + e^{-2\kappa_i(t-t_i)})}{2\kappa_i^2} \tag{3.11}$$

$$\int \beta_j dt = -\frac{e^{-2\kappa_j(t-t_j)}}{2\kappa_j} \tag{3.12}$$

If $i < h(t) - 1$ resp. $j < h(t) - 1$ these integrals can trivially computed by multiplying α_i resp. β_j by the interval length over which the integral is computed.

3.3. **Formula for $A(s,t)$.** Now we continue with the formulas for conditional expectation and variance 2.3. First of all we notice that

$$A(s,t) = \prod_{i=l(s)}^{h(t)-1} e^{-\kappa_i(t_{t,i+1}-t_{i,s})} = \prod_{i=l(s)}^{h(t)-1} \zeta_i \tag{3.13}$$

ζ_i is dependent on s if and only if $i = l(s)$. In this case

$$\int \zeta_i ds = \frac{1}{\kappa_i} e^{-\kappa_i(t_{t,i+1}-s)} \tag{3.14}$$

If $i > l(s)$ the integral can be computed trivially by multiplying ζ_i by the interval length over which the integral is computed.

3.4. **Formula for $E(x(t)|x(w))$.**

3.4.1. *The easy part.* The first term $A(w,t)x(w)$ in the conditional expectation can easily be computed with the result obtained so far. The second term is

$$\int_w^t A(s,t)(y(s) - \sigma_r(s)^2 G(s,T))ds \tag{3.15}$$

3.4.2. *First not so easy part of the integral.* Let's start with the integral over $A(s,t)y(s)$, which is

$$\sum_{k=l(w)}^{h(t)-1} \int_{t_{k,w}}^{t_{t,k+1}} \sum_{l=0}^{k} \alpha_l \left(\prod_{i=k}^{h(t)-1} \zeta_i \prod_{j=l+1}^{k} \beta_j \right) ds \tag{3.16}$$

We integrate each single summand, i.e. we fix k and l. ζ_i depends on s iff $i = k$. $beta$ depends on s iff $j = k$. α_l depends on s iff $l = k$.

Consider the case $l < k$ first. Then α_l does not depend on s. Amongst the factors in round brackets exactly ζ_k and β_k are depending on s. In essence we are left with computation of

$$\int \zeta_k \beta_k ds \tag{3.17}$$

which is

$$\int e^{-\kappa_k(t_{t,k+1}-s)} e^{-2\kappa_k(s-t_k)} ds \tag{3.18}$$

This again is explicitly

$$-\frac{1}{\kappa_k}e^{-\kappa_k s+\kappa_k(2t_k-t_{t,k+1})} \tag{3.19}$$

Now consider the case $l = k$. In this case exactly the factors $alpha_k$ and ζ_k depend on s. Note that β_k does not occur in the product in this case. We have therefore to evaluate

$$\int \zeta_k \alpha_k ds = \int e^{-\kappa_k(t_{t,k+1}-s)}\frac{\sigma_{r,k}^2}{2\kappa_k}\left[1-e^{-2\kappa_k(s-t_k)}\right]ds \tag{3.20}$$

This simplifies to

$$\frac{\sigma_{r,k}^2}{2\kappa_k}\int e^{-\kappa_k(t_{t,k+1}-s)} - e^{-\kappa_k s+\kappa_k(2t_k-t_{t,k+1})}ds \tag{3.21}$$

which is in explicit terms

$$\frac{\sigma_{r,k}^2}{2\kappa_k^2}\left(e^{-\kappa_k s+\kappa_k(2t_k-t_{t,k+1})} + e^{-\kappa_k(t_{t,k+1}-s)}\right) \tag{3.22}$$

3.4.3. *Second not so easy part of the integral.* Similarly the integral over $-A(s,t)\sigma_r(s)^2G(s,T)$ can be written

$$-\sum_{k=l(w)}^{h(t)-1}\sigma_k^2\int_{t_{k,w}}^{t_{t,k+1}}\sum_{l=k}^{h(T)-1}\eta_l\left(\prod_{i=k}^{h(t)-1}\zeta_i\prod_{j=k}^{l-1}\gamma_j\right)ds \tag{3.23}$$

As above, fix k and l. η_l is dependent on s iff $l = k$, ζ_i is dependent on s iff $i = k$, γ_j is dependent on s iff $j = k$.

Again we start wit the case $l > k$. As above we are left with $\int \zeta_k\gamma_k$, which is

$$\int e^{-\kappa_k(t_{t,k+1}-s)}e^{-\kappa_k(t_{k+1}-s)}ds \tag{3.24}$$

and explicitly

$$\frac{e^{2\kappa_k s-\kappa_k(t_{t,k+1}+t_{k+1})}}{2\kappa_k} \tag{3.25}$$

If on the other hand $l = k$, we face $\int \eta_k\zeta_k$ which can be computed as

$$\int e^{-\kappa_k(t_{t,k+1}-s)}\left(\frac{1-e^{-\kappa_k(t_{T,k+1}-s)}}{\kappa_k}\right)ds \tag{3.26}$$

and further

$$\frac{2e^{-\kappa_k(t_{t,k+1}-s)} - e^{2\kappa_k s-\kappa_k(t_{T,k+1}+t_{t,k+1})}}{2\kappa_k^2} \tag{3.27}$$

3.5. **Formula for** $\mathrm{Var}(x(t)|x(w))$**.** Finally we analyze the integral representing the conditional variance, which is

$$\int_w^t A(s,t)^2 \sigma_r(t)^2 ds \tag{3.28}$$

As before we write

$$\sum_{k=l(w)}^{h(t)-1} \sigma_{r,k}^2 \int_{t_{k,w}}^{t_{t,k+1}} \prod_{i=k}^{h(t)-1} \zeta_i^2 ds \tag{3.29}$$

For fixed k the term not covered yet is $\int \zeta_k^2 ds$, which is

$$\int e^{-2\kappa_k(t_{t,k+1}-s)} ds = \frac{e^{2\kappa_k(s-t_{t,k+1})}}{2\kappa_k} \tag{3.30}$$

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