

ONE FACTOR GAUSSIAN SHORT RATE MODEL IMPLEMENTATION

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ABSTRACT. We collect some results in Piterbarg, Interest Rate Modelling, needed for the implementation of a GSR model. We develop explicit formulas for piecewise constant volatility and reversion parameters under the forward measure.

1. MODEL

The short rate dynamics is given by

$$(1.1) \quad dr(t) = \kappa(t)(\theta(t) - r(t))dt + \sigma_r(t)dW(t)$$

under the risk neutral measure. κ, σ_r are piecewise constant. Setting $x(t) := r(t) - f(t, t)$ with $f(t, T)$ denoting the instantaneous forward rate observed at t for $T > t$, the dynamics can be rewritten

$$(1.2) \quad dx(t) = (y(t) - \kappa(t)x(t))dt + \sigma_r(t)dW(t)$$

with deterministic

$$(1.3) \quad y(t) = \int_0^t e^{-2 \int_u^t \kappa(s)ds} \sigma_r(u)^2 du$$

In the T -forward measure the dynamics becomes

$$(1.4) \quad dx(t) = (y(t) - \sigma_r(t)^2 G(t, T) - \kappa(t)x(t))dt + \sigma_r(t)dW^T(t)$$

with

$$(1.5) \quad G(t, t') = \int_t^{t'} e^{-\int_t^u \kappa(s)ds} du$$

This fits into the general treatment under 2.1 with

$$(1.6) \quad a(t) = -\kappa(t)$$

$$(1.7) \quad b(t) = y(t) - \sigma_r(t)^2 G(t, T)$$

$$(1.8) \quad c(t) = \sigma_r(t)$$

Zero bond prices can be expressed as follows

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$$(1.9) \quad P(t, t') = \frac{P(0, t')}{P(0, t)} e^{-x(t)G(t, t') - \frac{1}{2}y(t)G(t, t')^2}$$

2. BASIC SDE INTEGRATION

Consider the SDE

$$(2.1) \quad dx(t) = (a(t)x(t) + b(t))dt + c(t)dW(t)$$

with deterministic scalar functions a, b, c . The following is an explicit solution of 2.1.

$$(2.2) \quad x(t) = e^{\int_0^t a(u)du} \left(x(0) + \int_0^t e^{-\int_0^s a(u)du} b(s)ds + \int_0^t e^{-\int_0^s a(u)du} c(s)dW(s) \right)$$

That means that for $w < t$ $x(t)$ conditional on $x(w)$ is normally distributed with mean and variance given by

$$(2.3) \quad E(x(t)|x(w)) = A(w, t)x(w) + \int_w^t A(s, t)b(s)ds$$

$$(2.4) \quad \text{Var}(x(t)|x(w)) = \int_w^t A(s, t)^2 c(s)^2 ds$$

with short hand notation

$$(2.5) \quad A(s, t) = e^{\int_s^t a(u)du}$$

3. FORMULAS FOR THE PIECEWISE CONSTANT CASE

Under the assumption of piecewise constant κ, σ_r we are left with the computation of some integrals for which we derive closed form formulas here. We fix a grid $0 = t_0 < t_1 < \dots < t_n = T$ such that κ and σ_r are constant on each interval $[t_i, t_{i+1})$ and equal to $\kappa_i, \sigma_{r,i}$. We introduce the following notation: $l(t)$ denotes the largest index such that $t_{l(t)} \leq t$. Likewise $h(t)$ denotes the smallest index such that $t_{h(t)} \geq t$. Moreover we set $t_{i,s} := \max(t_i, s)$ and $t_{s,i} := \min(t_i, s)$.

3.1. Formula for $G(t, t')$. We start with 1.5, the formula for $G(t, t')$. The integrand is

$$(3.1) \quad e^{-\int_t^u \kappa(s)ds} = \prod_{i=l(t)}^{h(u)-1} e^{-\kappa_i \int_{t_{i,t}}^{t_{u,i+1}} ds} = \prod_{i=l(t)}^{h(u)-1} e^{-\kappa_i (t_{u,i+1} - t_{i,t})}$$

By this

$$(3.2) \quad G(t, t') = \sum_{i=l(t)}^{h(t')-1} \int_{t_{i,t}}^{t_{t',i+1}} \left(\prod_{j=l(t)}^{i-1} e^{-\kappa_j (t_{j+1} - t_{j,t})} \right) e^{-\kappa_i (u - t_{i,t})} du$$

which is

$$(3.3) \quad \sum_{i=l(t)}^{h(t')-1} \left(\frac{1 - e^{-\kappa_i(t_{t',i+1}-t_{i,t})}}{\kappa_i} \right) \prod_{j=l(t)}^{i-1} e^{-\kappa_j(t_{j+1}-t_{j,t})}$$

We abbreviate this by

$$(3.4) \quad G(t, t') = \sum_{i=l(t)}^{h(t')-1} \eta_i \prod_{j=l(t)}^{i-1} \gamma_j$$

γ_j is dependent on t if and only if $j = l(t)$. In this case $i > l(t)$ necessarily. η_i is dependent on t if and only if $i = l(t)$. In these cases

$$(3.5) \quad \int \gamma_j dt = \frac{e^{-\kappa_j(t_{j+1}-t)}}{\kappa_j}$$

$$(3.6) \quad \int \eta_i dt = \frac{t\kappa_i - e^{-\kappa_i(t_{t',i+1}-t)}}{\kappa_i^2}$$

If $j > l(t)$ resp. $i > l(t)$ these integrals can be computed trivially by multiplying γ_j resp. η_i by the interval length over which the integral is computed.

3.2. Formula for $y(t)$. We continue with 1.3. The integrand here is

$$(3.7) \quad e^{-2 \int_u^t \kappa(s) ds} \sigma_r(u)^2 = \prod_{i=l(u)}^{h(t)-1} \sigma_{r,i}^2 e^{-2\kappa_i(t_{t,i+1}-t_{i,u})}$$

We get

$$(3.8) \quad y(t) = \sum_{i=0}^{h(t)-1} \int_{t_i}^{t_{t,i+1}} \left(e^{-2\kappa_i(t_{t,i+1}-u)} \prod_{j=i+1}^{h(t)-1} \sigma_{r,j}^2 e^{-2\kappa_j(t_{t,j+1}-t_j)} \right) du$$

which is

$$(3.9) \quad \sum_{i=0}^{h(t)-1} \left(\frac{\sigma_{r,i}^2}{2\kappa_i} \left[1 - e^{-2\kappa_i(t_{t,i+1}-t_i)} \right] \prod_{j=i+1}^{h(t)-1} e^{-2\kappa_j(t_{t,j+1}-t_j)} \right)$$

and which we abbreviate by

$$(3.10) \quad y(t) = \sum_{i=0}^{h(t)-1} \left(\alpha_i \prod_{j=i+1}^{h(t)-1} \beta_j \right)$$

α_i resp. β_j is dependent on t if and only if $i = h(t) - 1$ resp. $j = h(t) - 1$. In the latter case $i < h(t) - 1$ necessarily. In these cases

$$(3.11) \quad \int \alpha_i dt = \frac{\sigma_{r,i}^2 (t\kappa_i + e^{-2\kappa_i(t-t_i)})}{2\kappa_i^2}$$

$$(3.12) \quad \int \beta_j dt = -\frac{e^{-2\kappa_j(t-t_j)}}{2\kappa_j}$$

If $i < h(t) - 1$ resp. $j < h(t) - 1$ these integrals can trivially be computed by multiplying α_i resp. β_j by the interval length over which the integral is computed.

3.3. Formula for $A(s, t)$. Now we continue with the formulas for conditional expectation and variance 2.3. First of all we notice that

$$(3.13) \quad A(s, t) = \prod_{i=l(s)}^{h(t)-1} e^{-\kappa_i(t_{t,i+1}-t_{i,s})} = \prod_{i=l(s)}^{h(t)-1} \zeta_i$$

ζ_i is dependent on s if and only if $i = l(s)$. In this case

$$(3.14) \quad \int \zeta_i ds = \frac{1}{\kappa_i} e^{-\kappa_i(t_{t,i+1}-s)}$$

If $i > l(s)$ the integral can be computed trivially by multiplying ζ_i by the interval length over which the integral is computed.

3.4. Formula for $E(x(t)|x(w))$.

3.4.1. The easy part. The first term $A(w, t)x(w)$ in the conditional expectation can easily be computed with the result obtained so far. The second term is

$$(3.15) \quad \int_w^t A(s, t)(y(s) - \sigma_r(s)^2 G(s, T)) ds$$

3.4.2. First not so easy part of the integral. Let's start with the integral over $A(s, t)y(s)$, which is

$$(3.16) \quad \sum_{k=l(w)}^{h(t)-1} \int_{t_{k,w}}^{t_{t,k+1}} \sum_{l=0}^k \alpha_l \left(\prod_{i=k}^{h(t)-1} \zeta_i \prod_{j=l+1}^k \beta_j \right) ds$$

We integrate each single summand, i.e. we fix k and l . ζ_i depends on s iff $i = k$. β_j depends on s iff $j = k$. α_l depends on s iff $l = k$.

Consider the case $l < k$ first. Then α_l does not depend on s . Amongst the factors in round brackets exactly ζ_k and β_k are depending on s . In essence we are left with computation of

$$(3.17) \quad \int \zeta_k \beta_k ds$$

which is

$$(3.18) \quad \int e^{-\kappa_k(t_{t,k+1}-s)} e^{-2\kappa_k(s-t_k)} ds$$

This again is explicitly

$$(3.19) \quad -\frac{1}{\kappa_k} e^{-\kappa_k s + \kappa_k (2t_k - t_{t,k+1})}$$

Now consider the case $l = k$. In this case exactly the factors α_k and ζ_k depend on s . Note that β_k does not occur in the product in this case. We have therefore to evaluate

$$(3.20) \quad \int \zeta_k \alpha_k ds = \int e^{-\kappa_k (t_{t,k+1} - s)} \frac{\sigma_{r,k}^2}{2\kappa_k} \left[1 - e^{-2\kappa_k (s - t_k)} \right] ds$$

This simplifies to

$$(3.21) \quad \frac{\sigma_{r,k}^2}{2\kappa_k} \int e^{-\kappa_k (t_{t,k+1} - s)} - e^{-\kappa_k s + \kappa_k (2t_k - t_{t,k+1})} ds$$

which is in explicit terms

$$(3.22) \quad \frac{\sigma_{r,k}^2}{2\kappa_k^2} \left(e^{-\kappa_k s + \kappa_k (2t_k - t_{t,k+1})} + e^{-\kappa_k (t_{t,k+1} - s)} \right)$$

3.4.3. *Second not so easy part of the integral.* Similarly the integral over $-A(s, t)\sigma_r(s)^2 G(s, T)$ can be written

$$(3.23) \quad - \sum_{k=l(w)}^{h(t)-1} \sigma_k^2 \int_{t_{k,w}}^{t_{t,k+1}} \sum_{l=k}^{h(T)-1} \eta_l \left(\prod_{i=k}^{h(t)-1} \zeta_i \prod_{j=k}^{l-1} \gamma_j \right) ds$$

As above, fix k and l . η_l is dependent on s iff $l = k$, ζ_i is dependent on s iff $i = k$, γ_j is dependent on s iff $j = k$.

Again we start with the case $l > k$. As above we are left with $\int \zeta_k \gamma_k$, which is

$$(3.24) \quad \int e^{-\kappa_k (t_{t,k+1} - s)} e^{-\kappa_k (t_{k+1} - s)} ds$$

and explicitly

$$(3.25) \quad \frac{e^{2\kappa_k s - \kappa_k (t_{t,k+1} + t_{k+1})}}{2\kappa_k}$$

If on the other hand $l = k$, we face $\int \eta_k \zeta_k$ which can be computed as

$$(3.26) \quad \int e^{-\kappa_k (t_{t,k+1} - s)} \left(\frac{1 - e^{-\kappa_k (t_{T,k+1} - s)}}{\kappa_k} \right) ds$$

and further

$$(3.27) \quad \frac{2e^{-\kappa_k (t_{t,k+1} - s)} - e^{2\kappa_k s - \kappa_k (t_{T,k+1} + t_{t,k+1})}}{2\kappa_k^2}$$

3.5. **Formula for $\text{Var}(x(t)|x(w))$.** Finally we analyze the integral representing the conditional variance, which is

$$(3.28) \quad \int_w^t A(s, t)^2 \sigma_r(t)^2 ds$$

As before we write

$$(3.29) \quad \sum_{k=l(w)}^{h(t)-1} \sigma_{r,k}^2 \int_{t_{k,w}}^{t_{k,k+1}} \prod_{i=k}^{h(t)-1} \zeta_i^2 ds$$

For fixed k the term not covered yet is $\int \zeta_k^2 ds$, which is

$$(3.30) \quad \int e^{-2\kappa_k(t_{k,k+1}-s)} ds = \frac{e^{2\kappa_k(s-t_{k,k+1})}}{2\kappa_k}$$

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