Valuation of Arithmetic Average of Fed Funds Rates and Construction of the US dollar Swap Yield Curve

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Abstract

Arithmetic averages of Fed Funds (FF) rates are paid on the FF leg of a FF-LIBOR basis swap, while the FF rates are paid with daily compounding in an overnight index swap. We consider here how to value the arithmetic average of FF rates and calculate convexity adjustment terms relative to daily compounded FF rates. FF-LIBOR basis swaps are now the critical calibration instruments for traders to construct the US dollar swap yield curve. We also show how it is constructed in practice.

1 Introduction

An interest rate swap (IRS), an interest rate basis swap (IRBS) and a cross currency basis swap (CCBS) are actively traded in the dealers' swap market. An IRS is the most fundamental interest rate product where fixed rates are exchanged for LIBOR rates. With an IRBS 2 different floating rates in the same currency are exchanged. Those 2 rates can be LIBORs with different tenors or different kinds of floating rates, for example, 3-month LIBOR vs 6-month LIBOR or the overnight (ON) rate vs LIBOR. A CCBS exchanges 3-month LIBORs in non-US dollar for US dollar 3-month LIBORs with the initial and final notional exchanges[1].

Since those swaps traded in the dealers' market are now fully collateralized backed up by the credit support annex (CSA) to an ISDA master agreement or through the settlements at LCH.clearnet, it has become common practice for traders to construct an OIS discounting curve and multiple forward curves for each LIBOR tenor (for example, see Bianchetti (2010) or Pallavicini and Tarenghi (2010)).

In the dealers' market ON rates are traded against LIBOR rates more actively in the form of an IRBS rather than against fixed rates in the form of an overnight indexed swap (OIS). Contrary to the popular belief, the OIS discounting curve is not constructed from quoted OIS rates. Rather

[1] US-dollar notional is reset at the FX spot rate at the start date of each interest period to mitigate counterparty risk.

it is constructed simultaneously with LIBOR forward curves from IRS rates and ON rate-LIBOR IRBS spreads.

On the ON-rate leg of an IRBS and an OIS, ON rates are usually paid with daily compounding in a single coupon. Although this daily compounding of ON rates is economically correct, Fed Funds (FF) rates are paid in the form of an arithmetic average without compounding in a US-dollar FF-LIBOR IRBS (see Credit Suisse Fixed Income Research (2010) for the details of FF rates convention used in the swap transactions).

This article shows how to differentiate the valuation of an arithmetic average of ON rates (AAON) from a daily compounded ON rates (DCON). We conclude that the AAON has convexity correction terms relative to the DCON. One correction term is "static" and the other one is "dynamic". The "dynamic" term can be further decomposed into 2 parts, of which one is very small compared with the other. This larger "dynamic" convexity correction term can be replicated by caplet/floorlet prices observed in the market. We also demonstrate how to construct the OIS discounting curve and LIBOR forward curves in US dollar, using FF-LIBOR IRSBs whose payoffs depend on the AAON.

2 Arithmetically averaged and daily compounded ON rates

In derivative products like swaps, ON rates are never paid on a daily basis. ON rates over one interest period are paid collectively in a single coupon in the form of an arithmetic average or on a daily compounded basis.

A variable rate linking to daily compounded ON rates (DCON) over an interest rate period $[T_s, T_e]$ is calculated as

$$R_c(T_s, T_e) = \frac{1}{\delta(T_s, T_e)} \left[\prod_{k=1}^{K} (1 + \delta_k C_k) - 1 \right]$$

while an arithmetic average of ON rates (AAON) is sampled as

$$R_a(T_s, T_e) = \frac{\sum_{k=1}^{K} \delta_k C_k}{\delta(T_s, T_e)}$$

where $\{C_k\}_{k=1,2,\cdots,K}$ are the effective ON rates[2] fixed in the interest period $[T_s, T_e]$ and $\{\delta_k\}_{k=1,2,\cdots,K}$ are the corresponding day count fractions. For an example of FF rates, $\delta_k = \frac{3}{360}$ for Friday and $\delta_k = \frac{1}{360}$ for other business days. $\delta(T_s, T_e)$ is the day count fraction for the whole interest period,

[2]Each effective ON rate is an average of ON rates traded at all brokers on a given day. The official fixing is usually announced on the next day by the central bank.

so that $\delta(T_s, T_e) = \sum_{k=1}^{K} \delta_k$. Hence the interest amounts over $[T_s, T_e]$ are calculated as

$$N * R_c(T_s, T_e) * \delta(T_s, T_e) = N * \left[\prod_{k=1}^{K} (1 + \delta_k C_k) - 1\right]$$

and

$$N * R_a(T_s, T_e) * \delta(T_s, T_e) = N * \sum_{k=1}^{K} \delta_k C_k$$

where N is a notional amount.

Interest rates are subject to the rate convention. The relationship between a simple compounded interest rate R and a continuously compounded interest rate r with the same day count convention is given by

$$1 + \delta R = e^{\delta r} \tag{1}$$

where δ is the day count fraction for the interest period. If we take R to be the ON rate, then the compounded amount over the interest period $[T_s, T_e]$ is calculated as

$$\prod_{k=1}^{K} (1 + \delta_k C_k) = e^{\sum_{k=1}^{K} \delta_k c_k} \tag{2}$$

where c_k is a continuously compounded ON rate which satisfies $1 + \delta C_k = e^{\delta c_k}$. Note that there is a limiting relationship which says:

$$\lim_{\delta \searrow 0} (1 + \delta R)^{1/\delta} = e^R$$

This means that for sufficiently small δ, the simple compounded interest rate reasonably approximates the continuously compounded rate. Since the ON rate has a small δ, (2) can be approximated by

$$\prod_{k=1}^{K} (1 + \delta_k C_k) = e^{\sum_{k=1}^{K} \delta_k C_k}$$

Hence AAON can be described as:

$$\begin{aligned} R_a(T_s, T_e) &= \frac{1}{\delta(T_s, T_e)} \log \prod_{k=1}^{K} (1 + \delta_k C_k) \\ &= \frac{1}{\delta(T_s, T_e)} \log (1 + \delta(T_s, T_e) R_c(T_s, T_e)) \end{aligned} \tag{3}$$

To see how precise this approximation is, we calculate the RHS and LHS of (3) with $\delta_k = \frac{1}{360}$ when $K = 360$ and $\delta(T_s, T_e) = 1$ in Table 1 and when $K = 90$ and $\delta(T_s, T_e) = 0.25$ in Table 2. Table 1 describes the results when the length of an interest period is 1 year and Table 2 does when the length is 3 month. In each table, we show the results when the sequence of ON rates starts with 10% and 1%. For each starting value C_1, we generated ON rates over the interest period by assuming they grows daily by 5bp, 1bp and 0bp. The approximation error depends on the length of the interest period, the starting ON rate and the growth rate, and the larger the those 3 parameters are, the larger the error becomes. Although the case where $R_1 = 10\%$ and $\Delta R = 5bp$ may be an extreme case, the approximation error in this case is 0.54bp with the annual interest period and 0.21bp with the quarterly interest period. Inspecting all cases in Table 1 and Table 2, and considering current realistic rate and volatility levels, we may conclude that AAON is reasonably approximated by the RHS of (3).

$C_1 = 10\%$								
$\Delta C = 5bp, R_c = 20.8882\%$			$\Delta C = 1bp, R_c = 12.5166\%$			$\Delta C = 0bp, R_c = 10.5156\%$		
R_a	approx.	diff	R_a	approx.	diff	R_a	approx.	diff
18.975%	18.9696%	$0.54bp$	11.795%	11.7931%	$0.20bp$	10%	9.9986%	$0.14bp$
$C_1 = 1\%$								
$\Delta C = 5bp, R_c = 10.4875\%$			$\Delta C = 1bp, R_c = 2.8343\%$			$\Delta C = 0bp, R_c = 1.005\%$		
R_a	approx.	diff	R_a	approx.	diff	R_a	approx	diff
9.975%	9.9733%	$0.17bp$	2.795%	2.7949%	$0.01bp$	1%	0.99999%	$0.00bp$

Table 1: The arithmetic average of ON rates and its approximation when $K = 360$

$C_1 = 10\%$								
$\Delta C = 5bp, R_c = 12.4115\%$			$\Delta C = 1bp, R_c = 10.5810\%$			$\Delta C = 0bp, R_c = 10.1246\%$		
R_a	approx.	diff	R_a	approx.	diff	R_a	approx.	diff
12.225%	12.2229%	$0.21bp$	10.445%	10.4435%	$0.15bp$	10%	9.9986%	$0.14bp$
$C_1 = 1\%$								
$\Delta C = 5bp, R_c = 3.2379\%$			$\Delta C = 1bp, R_c = 1.4476\%$			$\Delta C = 0bp, R_c = 1.0012\%$		
R_a	approx.	diff	R_a	approx.	diff	R_a	approx	diff
3.225%	3.2248%	$0.02bp$	1.445%	1.4450%	$0.00bp$	1%	0.99999%	$0.00bp$

Table 2: The arithmetic average of ON rates and its approximation when $K = 90$

(3) can be expanded at $R_c = 0$ for us to obtain

$$R_a(T_s, T_e) = R_c(T_s, T_e) - \frac{\delta(T_s, T_e) R_c(T_s, T_e)^2}{2} + \cdots \tag{4}$$

So one may say that the convexity correction of the AAON relative to the DCON is $\frac{\delta(T_s,T_e)R_c(T_s,T_e)^2}{2}$. These values are $50bp$ and $0.125bp$ respectively when R_c is 10% and $\delta(T_s, T_e) = 1$ and when R_c is 1% and $\delta(T_s, T_e) = 0.25$, the results which are consistent with the case of $\Delta C = 0bp$ in Table 1 and Table 2. However, this is not the end of the story of the convexity adjustment of the AAON.

3 Valuation of variable rates on the ON-rate leg

Fujii et al (2009) and Piterbarg (2010) have shown that since the interest rate accruing on the collateral account is the ON rate, the time t value of the collateralized European derivative $V(t)$ whose payoff at the maturity T is X can be written as

$$V(t) = E_t^Q \left[e^{-\int_t^T c(u)du} X \right] \tag{5}$$

for $t \leq T$, where $E_t^Q[\cdot]$ denotes the time t conditional expectation under the risk-neutral measure Q. Note the short rate form of the ON rate c replaces the risk-free short rate r in the usual pricing formula. This is good news because we do not observe the risk-free rate r in practice. The notion of "OIS discounting" comes from (5). We denote the time t price of the collateralized discount bond with the maturity T by

$$D(t;T) = E_t^Q \left[e^{-\int_t^T c(u)du} \right] \tag{6}$$

for $t \leq T$.

Denoting further the initial discount factor maturing at T by $D(T) \equiv D(0;T)$, the present value of a collateralized DCON over the interest period $[T_s, T_e]$ is given by[3],

$$\begin{aligned} V_c &= E^Q \left[e^{-\int_0^{T_e} c(u)du} \delta(T_s, T_e) R_c(T_s, T_e) \right] \\ &= E^Q \left[e^{-\int_0^{T_e} c(u)du} \left[\prod_{k=1}^{K} (1 + \delta_k C_k) - 1 \right] \right] \\ &= D(T_s) - D(T_e) \\ &= \delta(T_s, T_e) O(0; T_s, T_e) D(T_e) \end{aligned} \tag{7}$$

where

$$O(t; T_s, T_e) = \frac{1}{\delta(T_s, T_e)} \left[\frac{D(t; T_s)}{D(t; T_e)} - 1 \right], \; t \leq T_s \tag{8}$$

is a time t forward rate for the DCON and $E^Q[\cdot]$ denotes the initial unconditional expectation under the risk-neutral measure Q. The relation between c and C_k, $E_{t_k}^Q \left[e^{-\int_{t_k}^{t_{k+1}} c(u)du} \right] = \frac{1}{1+\delta_k C_k}$ is used from the 2nd to 3rd equation in (7). Note that (8) is exactly the same as the textbook forward LIBOR formula. Although the DCON is not completely fixed until the end of the interest period, for valuation purposes it can be treated as fixed at the OIS rate for the length of $[T_s, T_e]$ which is observed at the start date T_s. In this sense, $O(t; T_s, T_e)$ is called a time t forward OIS rate maturing at T_s as the spot OIS rate over $[T_s, T_e]$, in the same way that the forward LIBOR matures at T_s as the spot LIBOR[4].

[3] We used here the risk neutral expectation operator, but in the case of the DCON, the static replication argument also applies.

[4] More precisely, forward OIS matures on the trading date of the OIS while forward LIBOR matures on the

The present value of an AAON over the interest period $[T_s, T_e]$ is given by:

$$\begin{aligned}
V_a &= E^Q\left[e^{-\int_0^{T_e} c(u)du}\delta(T_s,T_e)R_a\right] \\
&= E^Q\left[e^{-\int_0^{T_e} c(u)du}\left[\sum_{k=1}^{K}\delta_k C_k\right]\right] \\
&\approx E^Q\left[e^{-\int_0^{T_e} c(u)du}\left[\log\prod_{k=1}^{K}(1+\delta_k C_k)\right]\right] \qquad (9)\\
&\approx E^Q\left[e^{-\int_0^{T_e} c(u)du}\int_{T_s}^{T_e} c(u)du\right] \\
&= D(T_e)E^{T_e}\left[\int_{T_s}^{T_e} c(u)du\right] \qquad (10)
\end{aligned}$$

where the expectation is taken under the payment date T_e-forward measure whose numeraire is the collateralized T_e-maturity discount bond $D(\cdot;T_e)$ in the last equation. From the 2nd to 3rd equation, the weighted sum of ON rates is approximated as (3). As we saw in the previous section, this approximation is reasonably good. From the 3rd to 4th equation, we approximate the log of the compounded amount by ON rates as its limit value, the integral of the short rate form of the ON rate, i.e.,

$$\lim_{\max\delta_k\searrow 0}\log\prod_{k=1}^{K}(1+\delta_k C_k) = \int_{T_s}^{T_e} c(u)du \qquad (11)$$

We will show later that most of the convexity correction of the AAON is explained without the last approximation. An AAON cannot be valued statically only with discount factors[5] as a DCON can be.

The time t forward rate for the AAON over $[T_s, T_e]$ can be defined as

$$O_a(t;T_s,T_e) = \frac{1}{\delta(T_s,T_e)}E_t^{T_e}\left[\int_{T_s}^{T_e} c(u)du\right],\ t \le T_s \qquad (12)$$

so that (10) is written as

$$V_a = \delta(T_s,T_e)O_a(0;T_s,T_e)D(T_e).$$

When the rates have no volatility, the ON short rate $c(u)$ at time u should be replaced by the initial instantaneous forward rate maturing at time u, $c(0;u)$. (10) in this case becomes

$$\begin{aligned}
\delta(T_s,T_e)O_a(0;T_s,T_e) &= \int_{T_s}^{T_e} c(0;u)du \\
&= \log\frac{D(T_s)}{D(T_e)} \\
&= \log(1+\delta(T_s,T_e)O(0;T_s,T_e)) \qquad (13)
\end{aligned}$$

LIBOR fixing date.

[5] As we will see later, the AAON can be valued almost statically with discount bonds and caplet/floorlet prices.

This is consistent with (3). Note also that we do not in fact need the final approximation of (11) to get (13). We plot $\log(1 + \delta(T_s, T_e)O(0; T_s, T_e))$ against $\delta(T_s, T_e)O(0; T_s, T_e)$ in Figure 1. It is clear from Figure 1 that there is convexity in the AAON relative to the DCON. Also we have

$$\delta(T_s, T_e)O(0; T_s, T_e) \geq \log(1 + \delta(T_s, T_e)O(0; T_s, T_e)), \tag{14}$$

with equality when $O = 0$.

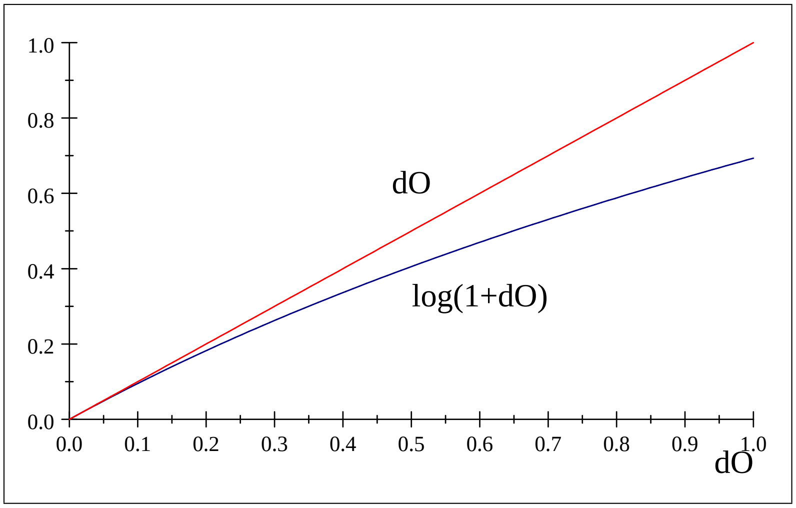


Figure 1: $\log(1 + \delta O)$ is plotted against δO.

Consider a strategy where an AAON is paid against its forward rate (12), and, as a delta hedge, the DCON is received against its forward rate $O(0; T_s, T_e)$. If the forward rate for the AAON is $\frac{1}{\delta(T_s,T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e))$ as in (13), the net payoff curve paid at T_e against the forward OIS rate is convex below and the minimum point is zero at the initial forward OIS rate. When the forward OIS rate moves in any direction, this strategy makes money. To avoid arbitrage,

$$O_a(0; T_s, T_e) < \frac{1}{\delta(T_s, T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e)) \tag{15}$$

should hold. Combining (14) with (15), we have:

$$O_a(0; T_s, T_e) < \frac{1}{\delta(T_s, T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e)) \leq O(0; T_s, T_e) \tag{16}$$

We conclude that the forward rate of the AAON over $[T_s, T_e]$ is smaller than that of the DCON over the same interest period by 2 convexity correction terms. The first convexity correction is static in the sense that it exists even when the forward OIS rate is not volatile at all, and its value is $O(0; T_s, T_e) - \frac{1}{\delta(T_s,T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e))$. This corresponds to the convexity correction in (4). The other convexity term is dynamic in the sense that it occurs due to the volatility of the forward OIS rate, and its value is $\frac{1}{\delta(T_s,T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e)) - O_a(0; T_s, T_e)$. We next evaluate this "dynamic" convexity correction term with the single-factor Hull White (HW) model and in a model-free way.

4 Single-factor Hull White Model

We apply the single-factor HW model to the short rate form of the ON rate, c. The dynamics of the ON short rate is described as

$$dc(t) = \left(\frac{\partial c(0;t)}{\partial t} + \varphi(t) - \lambda(t)(c(t) - c(0;t)) \right) dt + \sigma(t) dW^Q(t)$$

where

$$\varphi(t) = \int_0^t \sigma(u,t)^2 du$$

and $\sigma(t,T) = \sigma(u)e^{-\int_t^T \lambda(x)dx}$ is the normal instantaneous volatility of the time t forward rate with maturity T, namely, $c(t;T)$.

In a more practical form of the HW model, the one state variable Z satisfies the SDE

$$dZ(t) = -\lambda(t)Z(t)dt + \sigma(t)dW^Q(t), \; Z(0) = 0 \tag{17}$$

and $c(t;T)$ is written in terms of this state variable as

$$c(t;T) = c(0;T) + H(t,T) + e^{-\int_t^T \lambda(x)dx} Z(t) \tag{18}$$

Hence the spot rate is given by

$$c(t) = c(0;t) + H(t,t) + Z(t). \tag{19}$$

Here $H(t,T)$ represents rate-price convexity bias and can be written deterministically in terms of model parameters as

$$H(t,T) = \int_0^t \sigma(u,T)\nu(u,T)du \tag{20}$$

where $\nu(t,T)$ denotes the log-normal instantaneous volatility of $D(t;T)$ and is given by

$$\begin{aligned} \nu(t,T) &= \int_t^T \sigma(t,s)ds \\ &= \sigma(t) \int_t^T e^{-\int_t^s \lambda(x)dx} ds \end{aligned}$$

5 Pricing of arithmetic average of ON rates with Hull White model

We decompose the time T_e-forward value of the AAON, $E^{T_e}\left[\int_{T_s}^{T_e} c(u)du\right]$ in (10) into 2 parts and calculate them with the single-factor HW model.

$$E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} c(u)du\right] = E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} \left(c(u) - c(T_s;u)\right)du\right] + E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} c(T_s;u)du\right] \tag{21}$$

Using (18) and (19), the 1st term of the RHS in (21) becomes:

$$E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} \left(c(u) - c(T_s;u)\right)du\right] = \int_{T_s}^{T_e} \left(H(u,u) - H(T_s,u)\right)du + E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} \left(Z(u) - e^{-\int_{T_s}^{u}\lambda(x)dx} Z(T_s)\right)du\right] \tag{22}$$

Since

$$\begin{aligned} Z(u) &= e^{-\int_{T_s}^{u}\lambda(x)dx} Z(T_s) + \int_{T_s}^{u} \sigma(s,u)dW^Q(s) \\ &= e^{-\int_{T_e}^{u}\lambda(x)dx} Z(T_e) + \int_{T_e}^{u} \sigma(s,u)\left(dW^{T_e}(s) - \nu(s,T_e)ds\right) \end{aligned}$$

(22) further becomes:

$$E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} \left(c(u) - c(T_s;u)\right)du\right] = -\int_{T_s}^{T_e}\int_{T_s}^{u} \sigma(s,u)\left(\nu(s,T_e) - \nu(s,u)\right)dsdu \tag{23}$$

The 2nd term of the RHS in (21) can be written with the HW model as:

$$E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} c(T_s;u)du\right] = \int_{T_s}^{T_e} c(0;u)du + \int_{T_s}^{T_e} H(T_s;u)du + E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} e^{-\int_{T_s}^{u}\lambda(x)dx} Z(T_s)du\right] \tag{24}$$

Using

$$\begin{aligned} Z(T_s) &= \int_{0}^{T_s} \sigma(s,T_s)dW^Q(s) \\ &= \int_{0}^{T_s} \sigma(s,T_s)\left(dW^{T_e}(s) - \nu(s,T_e)ds\right), \end{aligned}$$

the 3rd term of the RHS in (24) can be explicitly calculated as:

$$E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} e^{-\int_{T_s}^{u}\lambda(x)dx} Z(T_s)du\right] = -\int_{T_s}^{T_e}\int_{0}^{T_s} \sigma(s,u)\nu(s,T_e)dsdu$$

Therefore, (24) becomes:

$$E^{\mathbf{T}_e}\left[\int_{T_s}^{T_e} c(T_s;u)du\right] = \log\frac{D(T_s)}{D(T_e)} - \int_{T_s}^{T_e}\int_0^{T_s}\sigma(s,u)\left(\nu(s,T_e)-\nu(s,u)\right)dsdu \tag{25}$$

From (23) and (25), we know that there are 2 convexity correction terms of the AAON from $\log\frac{D(T_s)}{D(T_e)}$ $(= \log(1+\delta O(0;T_s,T_e))$.

$$\text{convexity adj. } 1 = \int_{T_s}^{T_e}\int_0^{T_s}\sigma(s,u)\left(\nu(s,T_e)-\nu(s,u)\right)dsdu \tag{26}$$

and

$$\text{convexity adj. } 2 = \int_{T_s}^{T_e}\int_{T_s}^{u}\sigma(s,u)\left(\nu(s,T_e)-\nu(s,u)\right)dsdu \tag{27}$$

Convexity adjustment 1 represents the adjustment which comes from convex payoff of $\log\left(1+\delta(T_s,T_e)O(T_s;T_s,\right.$ against $\delta(T_s,T_e)O(T_s;T_s,T_e)$ at time T_s. Convexity adjustment 2 comes from the fact that ON rates are not actually fixed with their forward rates at the interest period start date and the payment of each ON rate is delayed without compounding. Note that the total convexity adjustment with the HW model is

$$\text{total convexity adj.} = \int_{T_s}^{T_e}\int_0^{u}\sigma(s,u)\left(\nu(s,T_e)-\nu(s,u)\right)dsdu \tag{28}$$

and that the 1st convexity adjustment is much larger than the 2nd one.

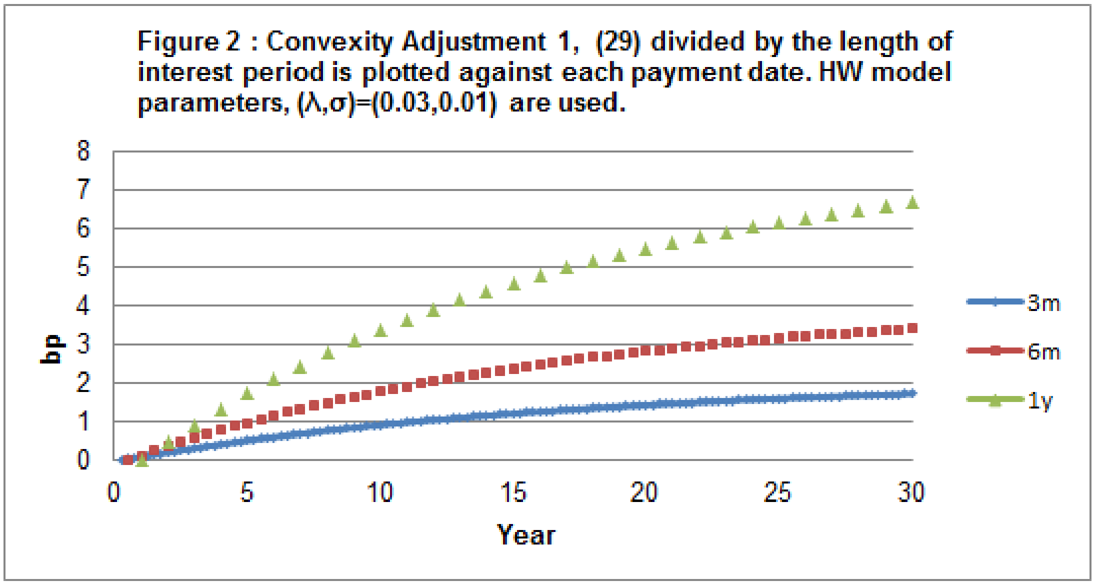
We show the numerical figures based on the plausible HW model volatility parameters roughly calibrated to the recent US dollar cap/floor market. For the constant mean reversion λ and short rate volatility σ, (26) and (27) become:

$$\text{convexity adj. } 1 = \frac{\sigma^2}{4\lambda^3}(1-e^{-2\lambda T_s})(1-e^{-\lambda(T_e-T_s)})^2 \tag{29}$$

and

$$\text{convexity adj. } 2 = \frac{\sigma^2}{2\lambda^2}\left[(T_e-T_s) - \frac{\left(1-e^{-\lambda(T_e-T_s)}\right)^2}{\lambda} - \frac{1-e^{-2\lambda(T_e-T_s)}}{2\lambda}\right] \tag{30}$$

With calibrated parameters $(\lambda,\sigma)=(0.03,0.01)$, convexity adjustments 1 and 2 are calculated for annual, semiannual and quarterly interest periods. The numbers are divided by length of the interest period (1, 0.5 and 0.25) to get annualized rates. Note that for constant HW model parameters, convexity adjustment 2 depends only on the length of interest period, $\delta = T_e - T_s$, no matter where the interest period is located. Convexity adjustment 2 is shown in Table 3 for each length of interest period. The longer the interest period is, the larger is convexity adjustment 2. This is because the payment of more ON rates is delayed longer when the interest period gets longer.



interest period	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1.0$
conv adj 2 / δ	0.01bp	0.04bp	0.16bp

Table 3: Convexity adjustment 2, (30) divided by the length of the interest period with $(\lambda, \sigma) = (0.03, 0.01)$.

Table 3 indicates that convexity adjustment 2 is small. Indeed, it is safe to be ignored when the length of the interest period is as small as 3 months or 6 months.

Figure 2 plots (29) divided by the interest period length (1, 0.5 and 0.25) for each payment date T_e. The reason that convexity adjustment 1 is larger for longer interest periods is that the convexity of $\delta^{-1}\log(1 + \delta O)$ is larger with respect to the OIS rate O. With the same convexity level, the larger the time to maturity of the forward OIS rate is, the more benefit one can get from the convexity.

We next plot in Figure 3 convexity adjustment 1 on the basis of the discount-factor weighted average up to T_M (x-axis in Figure 3) to get the same scale as a quoted basis swap spread. The numbers are calculated for each interest period $\delta = 1, 0.5$ and 0.25 as

$$\frac{\sum_{i=1}^{M} \mathrm{ConvAdj}(T_i)/\delta * D(T_i)}{\sum_{i=1}^{M} D(T_i)} \tag{31}$$

where $T_1 = \delta$, $T_i - T_{i-1} = \delta$, and $\mathrm{ConvAdj}(T_i)$ is an adjustment term (29) paid at T_i

Figure 3 shows that the calibrating instruments of the basis swaps where FF arithmetic average is exchanged for 3-month LIBOR have small but distinct convexity adjustment, say, the 10-year swap has about 0.5bp of convexity adjustment and the 30-year swap has about 1bp in terms of quoted basis swap spread. Considering the bid-offer spread of basis swaps of FF and 3-month

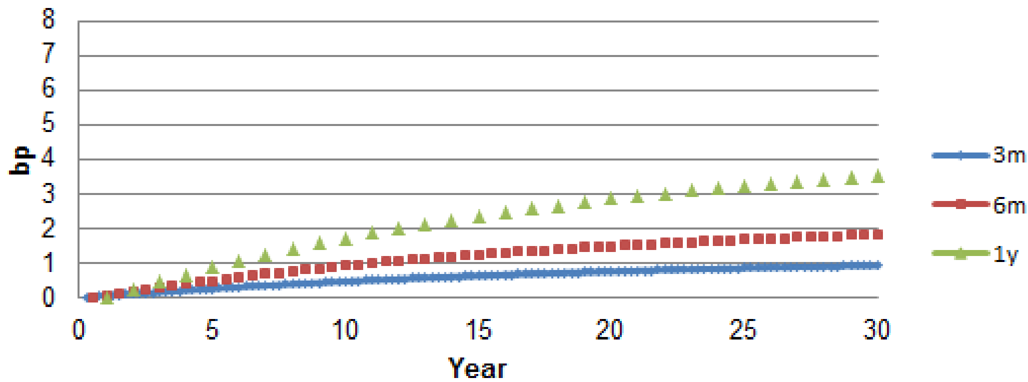


Figure 3: Convexity adjustment 1 on a quoted basis swap spread basis, (31) is plotted against maturity date, using US dollar discount factors as of June 15, 2011. HW model parameters, (λ,σ)=(0.03,0.01) are used.

LIBOR, it might be safe to ignore the convexity adjustment in the current market. However, when a swap trader may trade FF arithmetic averages with larger interest period than 3 months as a customer trade or when the volatility becomes larger than the current level, the convexity adjustment for the AAON may not be ignored.

The 1st term the RHS of (21) is so small as to be ignored. In this sense we actually did not have to use the limit approximation in (11), because

$$\log \prod_{k=1}^{K} (1 + \delta_k C_k(T_s)) = \int_{T_s}^{T_e} c(T_s; u)du$$

holds exactly in the forward rate version of (11), where $C_k(T_s)$ is the forward rate of C_k observed at T_s.

6 Model-free convexity adjustment for the arithmetic average of ON rates

We will show in this section that the 2nd term of the RHS in (21) can actually be calculated without assuming any models. The 2nd term of the RHS in (21) inside the expectation operator is given by

$$\begin{aligned}\int_t^T c(t; u)du &= -\log D(t; T) \\ &= \log (1 + \delta(t, T)O(t; t, T))\end{aligned}$$

where $O(t;t,T)$ is the spot OIS rate over the interest period $[t,T]$, and is given by

$$O(t;t,T) = \frac{1}{\delta(t,T)}\left(\frac{1}{D(t;T)} - 1\right) \tag{32}$$

Any twice differentiable payoff $f(O)$ can be re-written as (see, for example, Carr and Madan (2002))

$$f(O) = f(\kappa) + f'(\kappa)(O-\kappa) + \int_0^\kappa f''(K)(K-O)^+ dK + \int_\kappa^\infty f''(K)(O-K)^+ dK \tag{33}$$

Putting $f(O) = \log(\delta O + 1)$ and applying (33) to it, we have

$$\log(\delta O + 1) = \log(\delta\kappa + 1) + \frac{\delta}{\delta\kappa + 1}(O-\kappa) - \int_0^\kappa \frac{\delta^2}{(\delta K + 1)^2}(K-O)^+ dK - \int_\kappa^\infty \frac{\delta^2}{(\delta K + 1)^2}(O-K)^+ dK \tag{34}$$

Applying T-forward measure expectation to the both side of (34) and taking $\kappa = O(0;t,T)$, we have the 2nd term of the RHS in (21) as

$$E^{\mathbf{T}}\left[\int_t^T c(t;u)du\right] = \log(\delta O(0;t,T) + 1) - \int_0^{O(0;t,T)} \frac{\delta F_o(K)}{(\delta K + 1)^2} dK - \int_{O(0;t,T)}^\infty \frac{\delta C_o(K)}{(\delta K + 1)^2} dK \tag{35}$$

where $E^T[O(t;t,T)] = O(0;t,T)$ is used, $F_o(K)$ and $C_o(K)$ denote respectively time T-forward value of OIS foorlet and caplet with strike K and the interest period $[t,T]$. Comparing (35) with (25), it follows that the sum of the 2nd and 3rd terms of the RHS of (35) corresponds to convexity adjustment 1 in the previous section.

Since the market is not so matured that OIS caps/floors are traded, we will replace in (35) OIS caplets/floorlets with LIBOR caplets/floorlets which are traded actively in the market.

The time T-forward value of OIS caplet with strike K is given by:

$$\begin{aligned}
\delta(t,T)E^{\mathbf{T}}\left[(O(t;t,T) - K)^+\right] &= E^{\mathbf{T}}\left[\left(\frac{1}{D(t;T)} - 1 - \delta(t,T)K\right)^+\right] \\
&= E^{\mathbf{T}}\left[(D_{l-d}(t;T)(1 + \delta(t,T)L(t;t,T)) - 1 - \delta(t,T)K)^+\right] \qquad (36)
\end{aligned}$$

where we used the spot OIS formula (32) and spot LIBOR formula:

$$\begin{aligned}
L(t;t,T) &= \frac{1}{\delta(t,T)}\left(\frac{1}{D_l(t;T)} - 1\right) \\
&= \frac{1}{\delta(t,T)}\left(\frac{1}{D_{l-d}(t;T)D(t;T)} - 1\right)
\end{aligned}$$

where $D_l(\cdot;T)$ is the T-maturity LIBOR discount factor, $D_{l-d}(\cdot;T)$ is the LIBOR-discount rate spread discount factor maturing at T and $D_l(\cdot;T) = D_{l-d}(\cdot;T)D(\cdot;T)$. If we assume that spread discount factors are non-stochastic[6], i.e.,

$$D_{l-d}(t;T) = \frac{D_{l-d}(T)}{D_{l-d}(t)},$$

then (36) further becomes:

$$\delta E^{\mathbf{T}}\left[(O(t;t,T)-K)^+\right] = \frac{D_{l-d}(T)}{D_{l-d}(t)}\delta(t,T)E^{\mathbf{T}}\left[\left(L(t;t,T) - \frac{1}{\delta(t,T)}\left((1+\delta(t,T)K)\,\frac{D_{l-d}(t)}{D_{l-d}(T)} - 1\right)\right)^+\right]. \tag{37}$$

We know from this formula that 1 unit of OIS caplet with strike K is equivalent to $\frac{D_{l-d}(T)}{D_{l-d}(t)}$ unit of LIBOR caplet with strike $K' = \frac{1}{\delta}\left((1+\delta K)\,\frac{D_{l-d}(t)}{D_{l-d}(T)} - 1\right)$. Changing variable so that $K' = \frac{1}{\delta}\left((1+\delta K)\,\frac{D_{l-d}(t)}{D_{l-d}(T)} - 1\right)$ and substituting (37) to (35), we have

$$E^{\mathbf{T}}\left[\int_t^T c(t;u)du\right] = \log\left(\delta(t,T)O(0;t,T)+1\right) - \int_0^{L(0;t,T)} \frac{\delta F_l(K)}{(\delta K+1)^2}dK - \int_{L(0;t,T)}^{\infty} \frac{\delta C_l(K)}{(\delta K+1)^2}dK \tag{38}$$

where $F_l(K)$ and $C_l(K)$ are now respectively the time T-forward value of LIBOR foorlet and caplet with strike K and the interest period $[t,T]$. It is as if (35) holds with OIS caplet/floorlet replaced with LIBOR caplets/floorlets without any adjustments. (38) links convexity adjustments to the initial market prices of LIBOR cap/floor prices. While the convexity adjustment based on HW model cannot hit the cap/floor volatility smile, (38) reflects the market smile. (38) is the model-free, static replication formula of the value of the AAON with discount bonds and option prices.

7 US Dollar Yield Curve Construction

Traders construct swap yield curves from the liquid instruments in the dealers' market. Since fixed rates are exchanged for LIBORs in an IRS, swap traders prefer to trade ON rates against LIBORs in the form of an IRBS rather than against fixed rates in the form of an OIS in the interbank market. An OIS is rather a customer trade than an interbank trade. Contrary to popular belief, the OIS discounting curve is not constructed from quoted OIS rates. Rather, it is constructed simultaneously with LIBOR forward curves from IRS rates and ON rate-LIBOR IRBS spreads[7].

We hereafter focus on the US dollar swap market. Let $\{T_i\}_{i=0,1,2,\cdots}$ be a US dollar swap schedule of relevant dates at 3-month intervals with modified following date convention. Let an

[6] Practitioners often assume this.

[7] When inputs are IRS rates and IRBS spreads with exactly the same tenors, we can construct the OIS discounting curve only from synthetic OIS rates. However, IRSs are much more liquid than ON rate-LIBOR basis swaps and liquid swap maturity points for IRSBs are less than IRSs. It is not a good idea to first interpolate quoted basis spreads to fill in the missing basis spreads and, together with IRS rates, to obtain synthetic OIS rates.

initial time be time 0 and the swap spot date be T_0. US dollar discount factor maturing at T is denoted by:

$$D(T) \equiv D(0;T) = E^Q\left[e^{-\int_0^T c(u)du}\right]$$

where c is the short rate form of the Fed Funds rate.

We assume here for simplicity that IRSs and IRBSs are the only market instruments for constructing the yield curve[8]. Let S_N denote an annual-money IRS rate for the tenor of N year. Since the value of the IRS at the quoted IRS rate is zero, the IRS rate satisfies

$$S_N \sum_{i=1}^{N} \delta(T_{4(i-1)}, T_{4i}; act/360) D(T_{4i}) = \sum_{i=1}^{4N} \delta(T_{i-1}, T_i; act/360) L(0; T_{i-1}, T_i) D(T_i) \tag{39}$$

for each $N = 1, 2, \cdots$, where the collateralized 3-month forward LIBOR is given by

$$L(0; T_{i-1}, T_i) = \frac{1}{\delta(T_{i-1}, T_i; act/360)} \left(\frac{D_{3l-d}(T_{i-1}) D(T_{i-1})}{D_{3l-d}(T_i) D(T_i)} - 1 \right) \tag{40}$$

Here $D_{3l-d}(T)$ is the T-maturity spread discount factor of 3-month LIBOR minus the discount rate.

Let B_N denote a basis swap spread added to the averaged FF rates against 3-month LIBOR in the IRBS. The basis swap spread satisfies

$$\sum_{i=1}^{4N} \delta(T_{i-1}, T_i; act/360) L(0; T_{i-1}, T_i) D(T_i) = \sum_{i=1}^{4N} \delta(T_{i-1}, T_i; act/360) \left\{ O_a(0; T_{i-1}, T_i) + B_N \right\} D(T_i), \tag{41}$$

for each $N = 0.5, 0.75, 1, 2, \cdots$, where the collateralized 3-month forward "arithmetically averaged" OIS rate is given by

$$O_a(0; T_{i-1}, T_i) = \frac{1}{\delta(T_s, T_e; act/360)} \left(\log \frac{D(T_{i-1})}{D(T_i)} - \text{Convexity Adj}_i \right) \tag{42}$$

Here the convexity adjustment term can be calculated through some model as (28) or with cap/floor market prices as (38). Although theoretically the yield curve and the volatility curve are simultaneously determined through the convexity adjustment of the AAON, this term is calculated beforehand and is an constant input to the yield curve construction in practice. Convexity Adj $= 0$ may be justified because the interest period on the FF leg is 3 months and its convexity correction may be small in the current market as we saw in the HW model example.

From (39), (41) and with the proper interpolating method, we can solve for the discounting curve $\{D(t)\}_{t>0}$ and spread curve of 3-month LIBOR minus discount rate, $\{D_{3l-d}(t)\}_{t>0}$ subject to $D(0) = 1$ and $D_{3l-d}(0) = 1$. Note that $\{D(t)\}_{t>0}$ is a so-called OIS discounting curve and

[8] We ignore MM, LIBOR futures, and FRAs.

$\{D_{3l-d}(t)D(t)\}_{t>0}$ is a so-called 3-month LIBOR forward curve[9].

When traders further need a 6-month LIBOR forward curve, they can add 3-month LIBOR vs 6-month LIBOR IRBSs to the market instruments and solve for 6-month minus 3-month LIBOR spread discount factors $\{D_{6l-3l}(t)\}_{t>0}$ given $\{D(t)\}_{t>0}$ and $\{D_{3l-d}(t)\}_{t>0}$. In this process the usual bootstrapping method applies and the collateralized 6-month forward LIBOR is calculated as:

$$L(0; T_{i-2}, T_i) = \frac{1}{\delta(T_{i-2}, T_i; act/360)} \left(\frac{D_{6l-3l}(T_{i-2}) D_{3l-d}(T_{i-2}) D(T_{i-2})}{D_{6l-3l}(T_i) D_{3l-d}(T_i) D(T_i)} - 1 \right)$$

8 Conclusion

We have shown in this article that there exist convexity adjustment terms for the arithmetic average of ON rates against the daily compounded ON rates. This is a similar problem to the convexity correction of CMS rate against swap rate, the convexity correction of LIBOR in arrear against LIBOR in advance and the convexity correction of LIBOR future against FRA rate. We have calculated the convexity adjustment term both with a single-factor HW model and in a model-free way with market option prices.

For the US dollar curve construction, FF-LIBOR basis swaps are critical calibration instruments where the arithmetic average of FF is used as the variable rate on the FF leg. We showed how the US dollar yield curve is constructed in practice from IRSs and FF-LIBOR IRBSs. The US dollar yield curve is used to construct the curve in another currency when the swaps in that currency are collateralized by US dollar cash, which is plausible in the CSA agreements among major dealers in the local currency market as a result of making the most use of netting benefit. The "US dollar collateralized" yield curve in that currency should be constructed through quoted CCBS spreads so that the US dollar OIS curve is a funding curve. The Japanese yen swap market is a typical example for this.

[9] The reasons that we use spread discount factors $D_{3l-d}(t)$ instead of 3-month LIBOR discount factor $D_{3l}(t)$ are 2 fold. 1) The yield curve is much smoother when $D(t)$ and $D_{3l-d}(t)$ are interpolated separately. 2) Sensitivities are calculated correctly this way.

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