(Original article at http://en.wikipedia.org/wiki/Heston_model)

Basic Heston model

The basic Heston model assumes that S_t, the price of the asset, is determined by a stochastic process:[2]

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t^S$$

where ν_t, the instantaneous variance, is a CIR process:

$$d\nu_t = \kappa(\theta - \nu_t) \, dt + \xi \sqrt{\nu_t} \, dW_t^{\nu}$$

and dW_t^S, dW_t^{ν} are Wiener processes (i.e., random walks) with correlation ρ, or equivalently, with covariance ρ dt.

The parameters in the above equations represent the following:

- μ is the rate of return of the asset.
- θ is the **long variance**, or long run average price variance; as t tends to infinity, the expected value of ν_t tends to θ.
- κ is the rate at which ν_t reverts to θ.
- ξ is the **vol of vol**, or volatility of the volatility; as the name suggests, this determines the variance of ν_t.

If the parameters obey the following condition (known as the Feller condition) then the process ν_t is strictly positive [3]

$$2\kappa_t \theta \geq \xi^2 .$$

Extensions

In order to take into account all the features from the volatility surface, the Heston model may be a too rigid framework.[citation needed] It may be necessary to add degrees of freedom to the original model. A first straightforward extension is to allow the parameters to be time-dependent.[citation needed] The model dynamics are then written as:

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t^S .$$

Here ν_t, the instantaneous variance, is a time-dependent CIR process:

$$d\nu_t = \kappa_t(\theta_t - \nu_t) \, dt + \xi_t \sqrt{\nu_t} \, dW_t^{\nu}$$

and dW_t^S, dW_t^{ν} are Wiener processes (i.e., random walks) with correlation ρ. In order to retain model tractability, one may require parameters to be piecewise-constant.[citation needed]

Another approach is to add a second process of variance, independent of the first one.[citation needed]

$$\begin{aligned} dS_t &= \mu S_t \, dt + \sqrt{\nu_t^1} S_t \, dW_t^{S,1} + \sqrt{\nu_t^2} S_t \, dW_t^{S,2} \\ d\nu_t^1 &= \kappa^1(\theta^1 - \nu_t^1) \, dt + \xi^1 \sqrt{\nu_t^1} \, dW_t^{\nu^1} \\ d\nu_t^2 &= \kappa^2(\theta^2 - \nu_t^2) \, dt + \xi^2 \sqrt{\nu_t^2} \, dW_t^{\nu^2} \end{aligned}$$

A significant extension of Heston model to make both volatility and mean stochastic is given by Lin Chen (1996).[citation needed] In the Chen model the dynamics of the instantaneous interest rate are specified by

$$\begin{aligned} dr_t &= (\theta_t - r_t) \, dt + \sqrt{r_t} \, \sigma_t \, dW_t, \\ d\alpha_t &= (\zeta_t - \alpha_t) \, dt + \sqrt{\alpha_t} \, \sigma_t \, dW_t, \end{aligned}$$

$$d\sigma_t = (\beta_t - \sigma_t)\,dt + \sqrt{\sigma_t}\,\eta_t\,dW_t.$$

Risk-neutral measure

See Risk-neutral measure for the complete article

A fundamental concept in derivatives pricing is that of the Risk-neutral measure;[*citation needed*] this is explained in further depth in the above article. For our purposes, it is sufficient to note the following:

1. To price a derivative whose payoff is a function of one or more underlying assets, we evaluate the expected value of its discounted payoff under a risk-neutral measure.
2. A risk-neutral measure, also known as an equivalent martingale measure, is one which is equivalent to the real-world measure, and which is arbitrage-free: under such a measure, the discounted prices of each of the underlying assets is a martingale. See Girsanov's theorem.
3. In the Black-Scholes and Heston frameworks (where filtrations are generated from a linearly independent set of Wiener processes alone), any equivalent measure can be described in a very loose sense by adding a drift to each of the Wiener processes.
4. By selecting certain values for the drifts described above, we may obtain an equivalent measure which fulfills the arbitrage-free condition.

Consider a general situation where we have n underlying assets and a linearly independent set of m Wiener processes. The set of equivalent measures is isomorphic to $\mathbf{R}^m$, the space of possible drifts. Let us consider the set of equivalent martingale measures to be isomorphic to a manifold M embedded in $\mathbf{R}^m$; initially, consider the situation where we have no assets and M is isomorphic to $\mathbf{R}^m$.

Now let us consider each of the underlying assets as providing a constraint on the set of equivalent measures, as its expected discount process must be equal to a constant (namely, its initial value). By adding one asset at a time, we may consider each additional constraint as reducing the dimension of M by one dimension. Hence we can see that in the general situation described above, the dimension of the set of equivalent martingale measures is $m - n$.

In the Black-Scholes model, we have one asset and one Wiener process. The dimension of the set of equivalent martingale measures is zero; hence it can be shown that there is a single value for the drift, and thus a single risk-neutral measure, under which the discounted asset $e^{-\rho t}S_t$ will be a martingale.[*citation needed*]

In the Heston model, we still have one asset (volatility is not considered to be directly observable or tradeable in the market) but we now have two Wiener processes - the first in the Stochastic Differential Equation (SDE) for the asset and the second in the SDE for the stochastic volatility. Here, the dimension of the set of equivalent martingale measures is one; there is no unique risk-free measure.[*citation needed*]

This is of course problematic; while any of the risk-free measures may theoretically be used to price a derivative, it is likely that each of them will give a different price. In theory, however, only one of these risk-free measures would be compatible with the market prices of volatility-dependent options (for example, European calls, or more explicitly, variance swaps). Hence we could add a volatility-dependent asset;[*citation needed*] by doing so, we add an additional constraint, and thus choose a single risk-free measure which is compatible with the market. This measure may be used for pricing.

Implementation

A recent discussion of implementation of the Heston model is given in a paper by Kahl and Jäckel .[4]

Information about how to use the Fourier transform to value options is given in a paper by Carr and Madan.[5]

Extension of the Heston model with stochastic interest rates is given in the paper by Grzelak and Oosterlee.[6]

Derivation of closed-form option prices for time-dependent Heston model is presented in the paper by Gobet et al.[7]

Derivation of closed-form option prices for double Heston model are presented in papers by Christoffersen [8] and also Gauthier. [9]

References

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