Exotic FX Swap

Analytics

ver 1.0

Exotics Pricing	
Methodology Trading	
Credit Risk Pricing	

Details about the document

Project	Exotics Pricing
Version	ver 1.0
Date	January 24, 2002
Author(s)	I. Rigopoulos
Department	Methodology Trading
Organizational Unit	Credit Risk Pricing
Contact person	Ioannis Rigopoulos, 54673
Filename	ExoticFXSwapAnalytics.tex

Distribution list

Name	Position / Function	Department

List of amendments

Version	Date	Reason for and extent of changes	Author(s)
1.0	11 January 2002	Original version	I. Rigopoulos

Contents

1 A word on Symbols

The following rules apply in the usage of symbols in this article to represent various mathematical and financial quantities:

1. A unique letter or combination of letters (ligature) is reserved for representing a family of quantities that are "strongly" associated to each other. Sets of subscripts, superscripts and various accents — like $\tilde{}$, $\hat{}$,$'$ etc — serve to provide additional clarification when needed. For example, the letter R is used to represent various notions of the concept of interest rate. These include the short rate (i.e. instantaneous spot rate), the forward short rate, the spot and forward simply compounded rate and the spot and forward generic swap rate. Note that all these quantities are measured in the same unit, namely the $''\frac{1}{\mathrm{time}}$“. The fact is that these quantities are only "*apparently*“ different, since all of them reduce to special cases of the generic forward swap rate. The latter can be fully represented by: $R_t^{crv;\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}}$ where $''crv$“ is the curve associated with the rate, t is the observation time of the rate, $\widetilde{T}^0$ is the start of the associated swap, $\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}$ is the series of the coupon payment dates within the swap and $\mathtt{N}$ is the number of the coupons (periods) of the associated swap. Obviously this notation is not particularly pleasant to the untrained eye, so we could as well write: $R_t^{\widetilde{T}^0}$ or R_t or even R to mean the same thing when the rest is assumed known from the context.

2. There are cases where the same symbol may refer to two quantities which differ from each other in a more fundamental sense than the one described above. Take for example the R_t. There are three possible interpretations at hand:

   (a) R_t refers to some specific realization of the rate at time t. In other words it is a single number.

   (b) R_t refers to all possible realizations of the rate at time t. In other words it is a random variable.

   (c) R_t refers to all possible realizations of the rate at all possible times t. In other words it is a stochastic process.

   Nevertheless using the same symbol provides for enhanced readability. The appropriate interpretation should be implied by the context.

3. There might arise cases where we need to use separate symbols to distinguish between potentially different valuations of the fundamentally same quantity. For example, let CF be the symbol reserved for representing a generic cash flow (amount paid or received). In particular we write CF_t for a cash flow occurring at time t. Let's further suppose we are dealing with two separate cash flows, both occurring at the same time t, e.g. the first cash flow is paid in the form of a coupon by a treasury bond, the other is paid in the form of a dividend by a stock. How do we express them symbolically? There are three approaches:

   (a) By using some accent, i.e. CF_t for the first cash flow and CF_t' or $\widetilde{CF_t}$ for the second. This approach is very readable but only convenient for a small number of differing quantities.

   (b) By appending an integer index as subscript, i.e. $CF_{1;t}$ for the first cash flow and $CF_{2;t}$ for the second. This approach is necessary when dealing with a large number of quantities and/or when algebraic manipulation based on the index is needed (for example we could add 10 different cash flows by writing: $\sum_{i=1}^{10} CF_{i;t}$. This would not have been possible by either of the other two approaches.)

(c) By appending some label as subscript. Let's assume our treasury bond is called xyz and our stock is called is called XYZ. A natural expression would be: $CF_{\mathrm{xyz;t}}$ for the first cash flow and $CF_{\mathrm{XYZ;t}}$ for the second.

In the cases where an index or a label is needed, this is added always on the bottom right of the quantity symbol. This choice is inspired from the conventional usage in the case of time. Time is denoted by t. Different time instants are typically denoted by t_i.

4. Most of the quantities of interest in Finance are a function of one or more time variables. For example the forward swap rate $R_t^{crv;\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}}$ is a function of the $2+\mathtt{N}$ time variables $t,\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}$. Nevertheless a single time variable is more important because it serves as the time variable for the associated stochastic process. In the case of the forward swap rate this is t. We follow the convention to place this time variable at the bottom right as a subscript. So only a counting index/label or a time variable will be ever placed at the bottom right. Everything else goes to the top.

The following table lists all the symbols used in this article. In the left column several versions of each symbol are presented, according to the discussion above.

Table 1: Table of symbols

Symbol	Interpretation
$\mathbf{CP}$ or $\mathbf{CP}_i$ or $\mathbf{CP}_i^R$	i^{th} coupon received on a particular stream of cash flows associated with R
$\mathbf{N}$ or $\mathbf{N}_i$ or $\mathbf{N}_i^R$	Notional applied in the calculation of the i^{th} coupon associated with R
$\mathbf{m}$ or $\mathbf{m}_i$ or $\mathbf{m}_i^R$	Multiplier applied in the calculation of the i^{th} coupon associated with R
$\mathbf{s}$ or $\mathbf{s}_i$ or $\mathbf{s}_i^R$	Spread applied in the calculation of the i^{th} coupon associated with R
$\boldsymbol{I}$ or $\boldsymbol{I}_i$ or $\boldsymbol{I}_i^R$	Index applied in the calculation of the i^{th} coupon associated with R
$\boldsymbol{\tau}$ or $\boldsymbol{\tau}_i$ or $\boldsymbol{\tau}_i^R$	Accrual interval applied in the calculation of the i^{th} coupon associated with R
$\boldsymbol{T}$ or $\boldsymbol{T}_i$ or $\boldsymbol{T}_i^R$	Payment time of the i^{th} coupon associated with R
$\mathbf{F}$ or $\mathbf{F}^Q$ or $\mathbf{F}_i^Q$	Floor applied on the stochastic quantity "Q". i is counting index or identifying label
$\mathbf{C}$ or $\mathbf{C}^Q$ or $\mathbf{C}_i^Q$	Cap applied on the stochastic quantity "Q". i is counting index or identifying label
$\boldsymbol{r}$ or $\boldsymbol{r}_t$ or $\boldsymbol{r}_t^{t_0}$ or $\boldsymbol{r}_t^{t_0,t_1}$ or $\boldsymbol{r}_t^{crv;t_0,t_1}$	Forward libor rate at t for a libor starting at t_0, matured at t_1 and being associated with the curve "crv"
$\boldsymbol{R}$ or $\boldsymbol{R}_t$ or $\boldsymbol{R}_t^{\widetilde{T}^0}$ or $\boldsymbol{R}_t^{\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}}$ or $\boldsymbol{R}_t^{crv;\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}}$	Forward swap rate at t for a swap starting at $\widetilde{T}^0$, having coupon dates $\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}$ and being associated with the curve "crv"
$\boldsymbol{S}$ or $\boldsymbol{S}_t$ or $\boldsymbol{S}_t^{\widetilde{T}^0}$ or $\boldsymbol{S}_t^{\frac{ccy1}{ccy2};\widetilde{T}^0}$	Forward FX rate at t with maturity at $\widetilde{T}^0$ associated with the FX "$\frac{ccy1}{ccy2}$"
$\mathtt{N}$ or $\mathtt{N}^R$	Number of coupons associated with "R"
$\boldsymbol{t}$ or $\boldsymbol{t}_i$	Time. i is counting index or identifying label
$\mathfrak{D}()$ or $\mathfrak{D}(\boldsymbol{t})$ or $\mathfrak{D}^{\alpha}(\boldsymbol{t})$	Day-count fraction of the time interval t. α indicates the day-count convention used
$\widetilde{\boldsymbol{T}}$ or $\widetilde{\boldsymbol{T}}^R$	Set time of the forward swap rate "R"
$\widetilde{\boldsymbol{T}}^0$ or $\widetilde{\boldsymbol{T}}^{R;0}$	Start time of the forward swap rate "R"
$\widetilde{\boldsymbol{T}}^i$ or $\widetilde{\boldsymbol{T}}^{R;i}$, $i \geq 1$	Payment time of the $i^t h$ coupon associated with the forward swap rate "R"
$\widehat{\boldsymbol{T}}$ or $\widehat{\boldsymbol{T}}^S$	Set time of the forward FX rate "S"
$\widehat{\boldsymbol{T}}^0$ or $\widehat{\boldsymbol{T}}^{S;0}$	Maturity of the forward FX rate "S"
$\boldsymbol{\rho}$ or $\boldsymbol{\rho}_t$ or $\boldsymbol{\rho}_t^{\alpha,\beta}$	Correlation at t between α and β

$\boldsymbol{CF}$ or $\boldsymbol{CF_t}$ or $\boldsymbol{CF_{i;t}}$	Cash Flow occurring at time t. i is counting index or identifying label
$\boldsymbol{V}$ or $\boldsymbol{V_t}$ or $\boldsymbol{V_t^{\alpha}}$	Value at time t of some traded asset. α indicates the traded asset
$\boldsymbol{V}()$ or $\boldsymbol{V_{t_1}}(\boldsymbol{CF_{t_2}})$	Value at time t_1 of the cash flow CF_{t_2} with $t_1 \leq t_2$
$\boldsymbol{P}$ or $\boldsymbol{P_t}$ or $\boldsymbol{P_t^T}$ or $\boldsymbol{P_t^{crv;T}}$	Value at time t of a riskless bond with notional $= 1$ maturing at time T with $t \leq T$. crv is the indicator of the curve associated with the bond
$\mathfrak{P}$ or $\mathfrak{P}_t$ or $\mathfrak{P}_t^{T1}$ or $\mathfrak{P}_t^{T_1,T_2}$ or $\mathfrak{P}_t^{crv1;T_1,crv2;T_2}$	Value at time t of the ratio between two riskless bonds $T_1, crv1$ and $T_2, crv2$
$\boldsymbol{\mathcal{N}}$ or $\boldsymbol{\mathcal{N}_t}$ or $\boldsymbol{\mathcal{N}_t^{i;}}$	Numeraire at time t. i is economy indicator
$\boldsymbol{\mathcal{Q}}$ or $\boldsymbol{\mathcal{Q}^{\mathcal{N}}}$	Measure w.r.t. numeraire indicator $\mathcal{N}$
$\mathsf{E}(\ )$ or $\mathsf{E}_t^{\mathcal{Q}}(\boldsymbol{X})$	Expectation of the random variable X conditional on the maximal available information as of time t w.r.t. measure $\mathcal{Q}$
$\mathbf{PmtCcy}_i$	Payment currency. i is counting index or identifying label
$\mathbf{RatCcy}_i$	Rate currency. i is counting index or identifying label
$\mathbf{RepCcy}_i$	Report currency. i is counting index or identifying label
$\frac{\boldsymbol{ccy1}}{\boldsymbol{ccy2}}$	Defines the *type* of the exchange rate between two currencies. The associated spot FX rate $S_t^{\frac{ccy1}{ccy2};t}$ will equal the number of units of currency $ccy1$ needed to buy 1 unit of currency $ccy2$ at time t
$\mathfrak{F}$ or $\mathfrak{F}_t$ or $\mathfrak{F}_t^{\alpha}$	Information (i.e. σ–algebra) available by the processes indicated by α at time t
$\mathfrak{C}$ or $\mathfrak{C}^{\alpha}$	Correction factor applied on the random variable indicated by α
$\mathtt{MAX}(\boldsymbol{X},\boldsymbol{Y})$	Maximum of X and Y
$\mathtt{MIN}(\boldsymbol{X},\boldsymbol{Y})$	Minimum of X and Y
$\mathcal{N}()$ or $\mathcal{N}(\boldsymbol{x})$	Standard normal cumulative distribution
$\boldsymbol{\sigma}$ or $\boldsymbol{\sigma_t}$ or $\boldsymbol{\sigma_t^X}$	Percentage volatility of process X at time t
$\boldsymbol{\mu}$ or $\boldsymbol{\mu_t}$ or $\boldsymbol{\mu_t^X}$	Percentage drift of process X at time t
$\boldsymbol{w}$ or $\boldsymbol{w_t}$ or $\boldsymbol{w_t^i}$	Value of the i^{th} orthogonal component of a standard vector Wiener process at time t
$\boldsymbol{\xi}$ or $\boldsymbol{\xi_t}$ or $\boldsymbol{\xi_t^X}$	Conditional expectation of Radon-Nikodym derivative indicated by X at time t
$\boldsymbol{M}$ or $\boldsymbol{M_t}$ or $\boldsymbol{M_{i;t}}$	Martingale process. i is counting index or identifying label
$\boldsymbol{F_{i;t}}$ or $\boldsymbol{G_{i;t}}$	Ito process. i is counting index or identifying label

2 Introduction

This document is concerned with the analytical pricing of a swap consisted of two legs. Each leg describes a series of coupon payments $\mathrm{CP}_i \quad, i = 1, \ldots, n$. Each CP_i is paid at time T_i in some currency referred to as "Payment Currency", and is given by:

$$\mathrm{CP}_i = \mathrm{N}_i \left(\mathrm{m}_i I_i + \mathrm{s}_i \right) \tau_i \tag{2.1}$$

where

N_i = The notional at the start of the i^{th} accrual period in units of the "Payment Currency".
m_i = A constant — possibly time-dependent — called "multiplier".
I_i = The index applicable for the i^{th} coupon.
s_i = A constant — possibly time-dependent — called "spread".
τ_i = The length of the i^{th} accrual period in years, according to the respective daycount convention.

Additionally the index I_i can be restricted to vary in the interval $\left[\mathrm{F}^{I_i}, \mathrm{C}^{I_i}\right]$, where

F^{I_i} is some period dependent Index Floor level
C^{I_i} is some period dependent Index Cap level

We set $t = 0$ for today's time to simplify the notation. Let the value today of a cash flow CF_t occurring at any time $t \geq 0$ be $V_0(CF_t)$. Then due to the linearity of the value operator $V()$ and since $V_0(\mathrm{s}_i) = P_0^{T_i}\mathrm{s}_i$ we conclude:

$$V_0(\mathrm{CP}_i) = \mathrm{N}_i \left(\mathrm{m}_i V_0(I_i) + P_0^{T_i}\mathrm{s}_i \right) \tau_i \tag{2.2}$$

and

$$V_0(SwapLeg) = \sum_{i=1}^{\mathtt{N}^{SwapLeg}} V_0(\mathrm{CP}_i) = \sum_{i=1}^{\mathtt{N}^{SwapLeg}} \mathrm{N}_i \left(\mathrm{m}_i V_0(I_i) + P_0^{T_i}\mathrm{s}_i \right) \tau_i \tag{2.3}$$

where $\mathtt{N}^{SwapLeg}$ is the number of coupons in the swap leg "$SwapLeg$" and $P_0^{T_i}$ is the price today of a riskless bond on a 1 currency unit notional having maturity T_i.

The challenge in the above formulas is to calculate $V_0(I_i)$. The purpose of this document is to derive closed form formulas for this quantity, for different definitions of I_i. In the following section we start with the simplest case where the index is just some forward swap rate on the same currency as the payment currency and we derive the formula for the respective convexity correction. Each additional section will extend this basic setup by adding complexity to the definition of the index.

3 Vanilla CMS: Domestic Index Paid in Domestic currency.

This simplest case covers the so called CMS swaps. In a typical CMS swap the coupon of the floating leg is given by (2.1) where the index I_i is just a — generally forward — swap rate $R_{\widetilde{T}}$ that sets at $\widetilde{T}$ with accrual start at $\widetilde{T}^0$ and remaining coupon payment times given by $\widetilde{T}^1, \ldots, \widetilde{T}^{\mathtt{N}}$, where $\mathtt{N}$ is the number of the coupons. For example, $\mathtt{N} = 20$ for a 10-year semi-annual CMS rate. Using full notation, we could have written $R_{\widetilde{T}}^{\widetilde{T}^0, \widetilde{T}^1, \ldots, \widetilde{T}^{\mathtt{N}}}$ for this rate, but we prefer the simpler notation $R_{\widetilde{T}}$ since the context here is not ambiguous. Replacing $R_{\widetilde{T}}$ for the index I_i in (2.1) and (2.2) we get:

$$\mathbb{CP} = \mathrm{N}\left(\mathrm{m}R_{\widetilde{T}} + \mathrm{s}\right)\tau \tag{3.1}$$

$$V_0(\mathbb{CP}) = \mathrm{N}\left(\mathrm{m}V_0\left(R_{\widetilde{T}}\right) + P_0^T\,\mathrm{s}\right)\tau \tag{3.2}$$

where we dropped the coupon index i for simplicity.

The amount $\mathbb{CP}$ is paid out at time T as shown in figure 1.

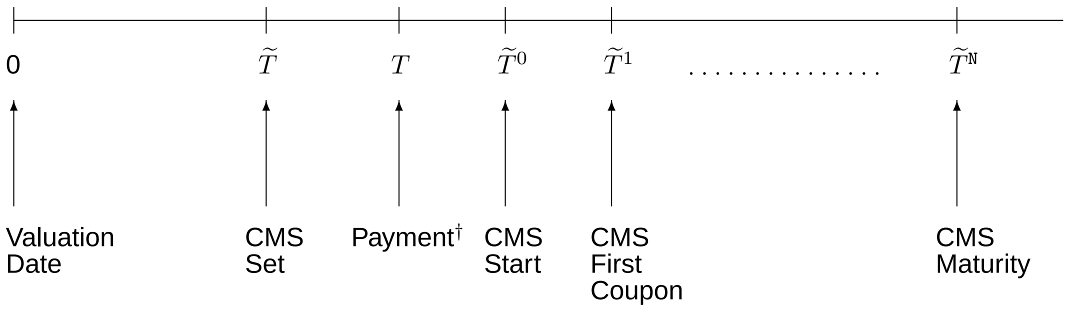

Fig. 1: Relevant dates for a coupon when the index is a CMS rate

In order to calculate $V_0\left(R_{\widetilde{T}}\right)$ we choose the bond P_t^T matured at T as numeraire. Let $\mathcal{Q}^P$ the associated measure. We know from the general theory[3, 4] that the assumption of no arbitrage opportunities implies the value of each traded asset divided by this numeraire is a martingale w.r.t. $\mathcal{Q}^P$. We also assume the market is complete $\Rightarrow$ there is some asset that replicates any given final cash flow $\Rightarrow$ there is some asset‡ whose value at T equals $R_{\widetilde{T}}$. $V_0\left(R_{\widetilde{T}}\right)$ is nothing else but the value of this asset as of 0. Writing V_t for the value of this asset at any time t we therefore have:

$$V_0 = V_0\left(R_{\widetilde{T}}\right) \qquad \text{and} \qquad V_T = R_{\widetilde{T}}$$

Since $\frac{V_t}{P_t^T}$ is a martingale in $\mathcal{Q}^P$ we get:

$$\frac{V_0}{P_0^T} = \mathsf{E}_0^{\mathcal{Q}^P}\left(\frac{V_T}{P_T^T}\right)$$

where $\mathsf{E}_t^{\mathcal{Q}}(X)$ is the expectation w.r.t. $\mathcal{Q}$ of the random variable X conditional on the information available at time t.

Using the two previous equations and the fact $P_T^T = 1$ we get:

$$V_0\left(R_{\widetilde{T}}\right) = P_0^T\,\mathsf{E}_0^{\mathcal{Q}^P}\left(R_{\widetilde{T}}\right) \tag{3.3}$$

† The placement order for the payment time shown here represents just one of the possibilities. In general it may occur at any time after $\widetilde{T}$. The rest of the times should be ordered as shown.

‡ the concept of "asset" encompasses self-financing trading strategies as well

It seems we only need to calculate $\mathsf{E}_0^{\mathcal{Q}^P}\left(R_{\widetilde{T}}\right)$. With a bit of foresight though, we would rather find the whole distribution function of $R_{\widetilde{T}}$ — note that $R_{\widetilde{T}}$ is a random variable adopted to the information $\mathfrak{F}_{\widetilde{T}}$. This is because later we will deal with caps and floors on the rate $R_{\widetilde{T}} \Longrightarrow$ we will have to calculate expressions like $\mathsf{E}_0^{\mathcal{Q}^P}\left(\mathtt{MAX}\left(R_{\widetilde{T}}, K\right)\right)$, where K can be any number. With some more foresight, we would better opt for the whole thing, i.e. the diffusion process of R_t , $0 \le t \le \widetilde{T}$. This "ultimate" knowledge will enable us later to calculate more exotic payoffs based on R_t, like knock-out barriers. Also note that knowing the distribution of R_t for each t is not enough. There are many different diffusions which lead to the same unconditional distributions.

Our aim is therefore to derive the Stochastic Differential Equation (SDE) of R_t , $0 \le t \le \widetilde{T}$ in the measure $\mathcal{Q}^P$.

Here is the result:

Result 3.1 [$\boldsymbol{SDE\ of\ Forward\ Swap\ Rate\ R_t\ in\ \mathcal{Q}^P}$]

$$\frac{1}{R_t}\frac{dR_t}{dt} = \mu^R dt + \sigma^R dw \qquad \textit{w.r.t. } \mathcal{Q}^P \quad \textit{for} \quad 0 \le t \le \widetilde{T} \tag{3.4}$$

where

$$\mu^R = \sigma^R\left(\sigma^R + \sigma^{\mathfrak{P}} - \sigma^{r^{\widetilde{T}^0,\widetilde{T}^{\mathtt{N}}}}\right) \tag{3.5}$$

σ^R , $\sigma^{r^{\widetilde{T}^0,\widetilde{T}^{\mathtt{N}}}}$ and $\sigma^{\mathfrak{P}}$ are all deterministic functions of time and $\sigma^{\mathfrak{P}}$ is approximated by the expression

$$\sigma^{\mathfrak{P}} \simeq \begin{cases} \left(1 - \frac{P_0^{\widetilde{T}^{\mathtt{N}}}}{P_0^T}\right)\sigma^{r^{T,\widetilde{T}^{\mathtt{N}}}} & \textit{if} \quad T \le \widetilde{T}^{\mathtt{N}} \\ \left(1 - \frac{P_0^T}{P_0^{\widetilde{T}^{\mathtt{N}}}}\right)\sigma^{r^{\widetilde{T}^{\mathtt{N}},T}} & \textit{if} \quad \widetilde{T}^{\mathtt{N}} \le T \end{cases} \tag{3.6}$$

R_t is a brief notation of $R_t^{\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}}$ which is the forward swap rate as seen at time t when the underlying swap starts at time $\widetilde{T}^0$ and pays coupons at times $\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}$. The order holds: $\widetilde{T} \le \widetilde{T}^0 < \widetilde{T}^1 < \ldots < \widetilde{T}^{\mathtt{N}}$.
σ^R is the percentage volatility of the forward swap rate process R_t. It can be inferred from the swaption market.[a]
$\sigma^{r^{t_1,t_2}}$ is the percentage volatility of the forward libor rate process r^{t_1,t_2}. It can be inferred from the cap market.[b]
$\sigma^{\mathfrak{P}}$ is the percentage volatility of the bond ratio process $\frac{P_t^T}{P_t^{\widetilde{T}^{\mathtt{N}}}}$.
$\mathcal{Q}^P$ is the equivalent martingale measure associated with the numeraire being the bond price P_t^T maturing at the coupon payment time T.
Also note the assumptions 3.2, 3.3, 3.4 and 3.5 apply.

[a] Section 3.1
[b] Section 3.2

As a first application of the *result* 3.1 we can calculate explicitly the expectation in 3.3:

$$V_0\left(R_{\widetilde{T}}\right) = P_0^T R_0 e^{\int_0^{\widetilde{T}} \mu_u^R\, du} \tag{3.7}$$

This in turn can be set in 3.2 to get the price today of the whole coupon.

By comparing (3.7) with the well known result $V_0\Big(r_{\widetilde{T}}\Big) = P_0^T r_0$ in the special case when $r_{\widetilde{T}}$ is an amount equal to the spot at $\widetilde{T}$ libor rate from $\widetilde{T}$ to T paid at T — i.e. when $r_{\widetilde{T}} \equiv R_{\widetilde{T}}^{\widetilde{T},T} \equiv r_{\widetilde{T}}^{\widetilde{T},T}$ —†,and where by r_0 we mean the forward rate $r_0^{\widetilde{T},T}$, we derive the following result:

> **Result 3.2** [$\boldsymbol{CMS\ Convexity\ Correction\ for\ Cash\ Flows\ linear\ in\ R_{\widetilde{T}}}$]
>
> *When the coupon payment* $\mathbb{CP}$ *at time* T *is linear on* $R_{\widetilde{T}} \equiv R_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1,\ldots\widetilde{T}^{\mathtt{N}}}$, $\widetilde{T} \le T$ *and* $\widetilde{T} \le \widetilde{T}^0 < \widetilde{T}^1 < \ldots < \widetilde{T}^{\mathtt{N}}$, *i.e. if* $\mathbb{CP} = aR_{\widetilde{T}} + b$, *with* a, b *constants, then its value today is:*
>
> $$V_0(\mathbb{CP}) = P_0^T\left(aR_0\mathfrak{C}^R + b\right) \tag{3.8}$$
>
> *where* $\mathfrak{C}^R$ *is some correction factor, typically called the "convexity correction factor", given by:*
>
> $$\mathfrak{C}^R = e^{\int_0^{\widetilde{T}}\mu_u^R\,du} \tag{3.9}$$
>
> *where* μ^R *is given by (3.5).*
> *We remind* P_0^T *is today's price of the riskless bond with maturity* T *and* $R_0 \equiv R_0^{\widetilde{T}^0,\widetilde{T}^1,\ldots\widetilde{T}^{\mathtt{N}}}$ *is the forward swap rate as seen from today.*

Finally we may want to value some generic — possibly non-linear in $R_{\widetilde{T}}$ — cash-flow CF_T paid out at time T. The following result holds:

> **Result 3.3** [$\boldsymbol{Valuation\ Expression\ for\ Cash\ Flows\ which\ are\ functions\ of\ R_{\widetilde{T}}}$]
>
> *Let* CF_T *some cash-flow paid at time* T, *which is a — possibly non-linear — function of* $R_{\widetilde{T}} \equiv R_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1,\ldots\widetilde{T}^{\mathtt{N}}}$, $\widetilde{T} \le T$ *and* $\widetilde{T} \le \widetilde{T}^0 < \widetilde{T}^1 < \ldots < \widetilde{T}^{\mathtt{N}}$, *i.e.* $CF_T \underset{def}{=} f(R_{\widetilde{T}})$. *Then its value today is given by:*
>
> $$V_0\big(CF_T\big) = P_0^T\left[\mathsf{E}_0^{\mathcal{Q}^P}\Big(f(R_{\widetilde{T}})\Big)\right] \tag{3.10}$$
>
> *where* P_t^T *is the price at* t *of the riskless bond with maturity* T *and* $\mathcal{Q}^P$ *is the equivalent martingale measure associated with* P_t^T.
> *The expectation can be in principle calculated by making use of the SDE of* R_t , $t \le \widetilde{T}$ *w.r.t.* $\mathcal{Q}^P$ *according to result 3.1.*

Proof of ***result*** **3.1**

By the definition of R_t it follows that R_t can be written in terms of bond prices as:

$$R_t = \frac{P_t^{\widetilde{T}^0} - P_t^{\widetilde{T}^{\mathtt{N}}}}{\sum_{i=1}^{\mathtt{N}}\widetilde{\tau}_i P_t^{\widetilde{T}^i}}$$

where $\widetilde{T}^i$, $i = 0, \ldots, \mathtt{N}$, are as shown in figure 1.
and $\widetilde{\tau}_i$ refers to the daycount fraction associated with the accrual interval $(\widetilde{T}^{i-1}, \widetilde{T}^i)$.

† the symbol "r" is specially reserved for libor rates.

Consider the measure $\mathcal{Q}^R$ associated with the numeraire $\sum_{i=1}^{\mathrm{N}} \widetilde{\tau}_i P_t^{\widetilde{T}^i}$. The expression above implies R_t is martingale w.r.t. $\mathcal{Q}^R$. Further on, we need to make the following assumption:

Assumption 3.1 [$\boldsymbol{Positive\ Rates\ Model}$] *The bond prices are positive and monotonically decreasing with maturity, i.e.* $P_t^{t_1} > P_t^{t_2}$ *for all* $t \le t_1 < t_2$

This results in $R_t > 0$ which in turn implies R_t is an exponential martingale[6, Page 73] and therefore obeys the following SDE in $\mathcal{Q}^R$:

$$\frac{1}{R_t}\frac{dR_t}{dt} = \sigma^R dw \qquad \text{w.r.t. } \mathcal{Q}^R \quad \text{for} \quad 0 \le t \le \widetilde{T} \tag{3.11}$$

Here w is the standard Wiener process†

and σ^R is some stochastic process sufficiently constrained so that R_t is indeed a martingale. A sufficient condition is the so-called Novikov condition:[5, Theorem 6.1]

$$\mathsf{E}^{\mathcal{Q}^R}\left(e^{\frac{1}{2}\int_0^{\widetilde{T}}\left|\sigma^R\right|^2 dt}\right) < \infty$$

Below we will restrict our study to the so called "lognormal" model, according to which the swap rate R_t for fixed t is lognormally distributed. This is equivalent to imposing the additional assumption below:

Assumption 3.2 [$\boldsymbol{Lognormal\ Swap\ Rate\ Model}$] *The percentage volatility* σ^R *of the forward swap rate* R_t *is a deterministic bounded function of time.*

Obviously the Novikov condition is then satisfied $\Longrightarrow$ the rate R_t is well defined as an exponential martingale.

The next step is to use Girsanov's theorem[2] which in this case implies that in the measure $\mathcal{Q}^P$ the process R_t will still keep the same percentage volatility σ^R but will acquire some drift μ^R — itself an Ito process. So we may write for the SDE of R_t in $\mathcal{Q}^P$:

$$\frac{1}{R_t}\frac{dR_t}{dt} = \mu^R dt + \sigma^R dw \qquad \text{w.r.t. } \mathcal{Q}^P \quad \text{for} \quad 0 \le t \le \widetilde{T} \tag{3.12}$$

Later we will show that the deterministic function of time σ^R can be inferred from the market prices of european swaptions. So we only need to calculate the drift μ^R. One way to proceed is by calculating the conditional expectation of R_t in $\mathcal{Q}^P$ by means of the change of measure formula:

$$\mathsf{E}_t^{\mathcal{Q}^P}\left(R_{t'}\right) = \frac{1}{\xi_t}\mathsf{E}_t^{\mathcal{Q}^R}\left(R_{t'}\xi_{t'}\right) \qquad \text{for} \quad 0 \le t \le t' \le \widetilde{T} \tag{3.13}$$

where $\xi_t = \mathsf{E}_t^{\mathcal{Q}^R}\left(\frac{d\mathcal{Q}^P}{d\mathcal{Q}^R}\right)$ and $\frac{d\mathcal{Q}^P}{d\mathcal{Q}^R}$ is the Radon-Nikodym derivative of $\mathcal{Q}^P$ w.r.t. $\mathcal{Q}^R$.

It is a well known fact — for example [1, Page 191] — that ξ_t equals the ratio of the respective normalized numeraires. The normalized numeraire at time t associated with $\mathcal{Q}^P$ is $\frac{P_t^T}{P_0^T}$ and the one associated with $\mathcal{Q}^R$ is $\frac{\sum_{i=1}^{\mathrm{N}} \widetilde{\tau}_i P_t^{\widetilde{T}^i}}{\sum_{i=1}^{\mathrm{N}} \widetilde{\tau}_i P_0^{\widetilde{T}^i}}$. So we have:

† as shown in [6] the Wiener process and the associated volatility are generally k-dimensional $\Longrightarrow$ the expression $\sigma^R dw$ is actually a shorthand of $\sum_{i=1}^k \sigma^j dw^j$. But we may replace $\sum_{i=1}^k \sigma^j dw^j$ with $\sigma^{eff} dw^{eff}$ where σ^{eff} and w^{eff} are both 1-dimensional. To achieve completeness we need of'course to restrict the associated filtration to the one generated by w^{eff} — which is smaller than the original product filtration generated by w^j , $i = 1, \ldots, k$

$$\xi_t = \frac{\frac{P_t^T}{P_0^T}}{\frac{\sum_{i=1}^{N} \widetilde{\tau}_i P_t^{\widetilde{T}^i}}{\sum_{i=1}^{N} \widetilde{\tau}_i P_0^{\widetilde{T}^i}}} = \frac{P_t^T}{\sum_{i=1}^{N} \widetilde{\tau}_i P_t^{\widetilde{T}^i}} \frac{\sum_{i=1}^{N} \widetilde{\tau}_i P_0^{\widetilde{T}^i}}{P_0^T}$$

When we replace ξ_t in (3.13) with the expression above, the factor $\frac{\sum_{i=1}^{N} \widetilde{\tau}_i P_0^{\widetilde{T}^i}}{P_0^T}$ drops out and we get:

$$\mathsf{E}_t^{\mathcal{Q}^P}\left(R_{t'}\right) = \frac{\sum_{i=1}^{N} \widetilde{\tau}_i P_t^{\widetilde{T}^i}}{P_t^T} \mathsf{E}_t^{\mathcal{Q}^R}\left(R_{t'} \frac{P_{t'}^T}{\sum_{i=1}^{N} \widetilde{\tau}_i P_{t'}^{\widetilde{T}^i}}\right) = \frac{1}{M_t} \mathsf{E}_t^{\mathcal{Q}^R}\left(R_{t'} M_{t'}\right) \tag{3.14}$$

where in the last step we replaced the fraction outside and inside the expectation by M_t and $M_{t'}$ respectively, whose process is defined by:

$$M_t \underset{def}{=} \frac{P_t^T}{\sum_{i=1}^{N} \widetilde{\tau}_i P_t^{\widetilde{T}^i}}$$

Now, because of (3.12) we have:

$$\mathsf{E}_t^{\mathcal{Q}^P}\left(R_{t'}\right) = R_t e^{\int_t^{t'} \mu_u \, du} \tag{3.15}$$

On the other hand, the positivity of $R_t M_t$ implies:

$$\mathsf{E}_t^{\mathcal{Q}^R}\left(R_{t'} M_{t'}\right) = R_t M_t e^{\int_t^{t'} \mu_u^{RM} du} \tag{3.16}$$

where μ^{RM} is the percentage volatility of $R_t M_t$.

The *assumption* (3.3) implies:

$$\mu^{RM} = \sigma^R \sigma^M \tag{3.17}$$

where σ^M is the percentage volatility of M_t which since it is a positive martingale w.r.t. $\mathcal{Q}^R$, it will satisfy:

$$\frac{1}{M_t}\frac{dM_t}{dt} = \sigma^M dt \qquad \text{w.r.t. } \mathcal{Q}^R \quad \text{for} \quad 0 \le t \le \widetilde{T} \tag{3.18}$$

Now we replace the expectations in (3.14) from (3.15) and (3.16) to get:

$$R_t e^{\int_t^{t'} \mu_u \, du} = R_t e^{\int_t^{t'} \mu_u^{RM} du} \Rightarrow \int_t^{t'} \mu_u^R du = \int_t^{t'} \mu_u^{RM} du$$

and since t' is arbitrary we conclude $\mu^R = \mu^{RM}$ and by using (3.17) :

$$\mu^R = \sigma^R \sigma^M \tag{3.19}$$

In order to find σ^M we rewrite M_t as:

$$M_t = \frac{P_t^T}{\sum_{i=1}^{N} \widetilde{\tau}_i P_t^{\widetilde{T}^i}} = \frac{P_t^T}{\sum_{i=1}^{N} \widetilde{\tau}_i P_t^{\widetilde{T}^i}} \frac{P_t^{\widetilde{T}^0} - P_t^{\widetilde{T}^N}}{P_t^{\widetilde{T}^0} - P_t^{\widetilde{T}^N}} = R_t \frac{P_t^T}{P_t^{\widetilde{T}^0} - P_t^{\widetilde{T}^N}} = R_t F_t \tag{3.20}$$

where

$$F_t \underset{def}{=} \frac{P_t^T}{P_t^{\widetilde{T}^0} - P_t^{\widetilde{T}^N}}$$

is some positive Ito process.

In order to simplify the subsequent formulas we will make the following assumption:

Assumption 3.3 [$\boldsymbol{One\ Factor\ Model}$] *All bond prices are driven by the same factor. In other words all yield curve quantities (including swap rates and bond prices) are perfectly correlated.*

Then we can write the SDE of F_t as follows:

$$\frac{1}{F_t}\frac{dF_t}{dt} = \mu^F dt + \sigma^F dw \qquad \text{w.r.t. } \mathcal{Q}^R \quad \text{for} \quad 0 \le t \le \widetilde{T} \tag{3.21}$$

where dw is the same as the one appearing in the SDE for R_t in (3.11), because of the *assumption* 3.3.

Now from (3.11),(3.18),(3.20) and (3.21) we conclude:

$$\sigma^M = \sigma^R + \sigma^F \tag{3.22}$$

It remains to calculate σ^F. We rewrite first F_t in terms of the forward libor rate $r_t^{\widetilde{T}^0,\widetilde{T}^N}$, which is the libor rate associated with the period $[\widetilde{T}^0, \widetilde{T}^N]$. For notational simplicity we set $r_t = r_t^{\widetilde{T}^0,\widetilde{T}^N}$ below. By the definition of the simply compounded forward libor rate we have:

$$r_t \equiv r_t^{\widetilde{T}^0,\widetilde{T}^N} \underset{def}{=} \frac{P_t^{\widetilde{T}^0} - P_t^{\widetilde{T}^N}}{\left(\widetilde{T}^N - \widetilde{T}^0\right) P_t^{\widetilde{T}^N}}$$

Plugging this in the definition of F_t we get:

$$F_t = \frac{1}{\widetilde{T}^N - \widetilde{T}^0}\frac{1}{r_t}\frac{P_t^T}{P_t^{\widetilde{T}^N}} \tag{3.23}$$

Now we define the positive Ito process $\mathfrak{P}_t^{T,\widetilde{T}^N}$ or simply $\mathfrak{P}_t$ by:

$$\mathfrak{P}_t \equiv \mathfrak{P}_t^{T,\widetilde{T}^N} \underset{def}{=} \frac{P_t^T}{P_t^{\widetilde{T}^N}} \tag{3.24}$$

Note that $\mathfrak{P}_t$ is just a forward discount factor if $T > TM$. Otherwise it equals the inverse of a discount factor.

We can now write (3.23) in terms of $\mathfrak{P}_t$:

$$F_t = \frac{1}{\widetilde{T}^N - \widetilde{T}^0}\frac{1}{r_t}\mathfrak{P}_t \tag{3.25}$$

Based on our assumptions so far, r_t and $\mathfrak{P}_t$ are perfectly correlated positive Ito processes. Let $\sigma^{r^{\widetilde{T}^0,\widetilde{T}^N}}$ and $\sigma^{\mathfrak{P}}$ the corresponding percentage volatilities. Then (3.25) implies:

$$\sigma^F = \sigma^{\mathfrak{P}} - \sigma^{r^{\widetilde{T}^0,\widetilde{T}^{\mathtt{N}}}} \tag{3.26}$$

It remains to calculate $\sigma^{\mathfrak{P}}$.

We consider 3 cases:

Case 1 $T < \widetilde{T}^{\mathtt{N}}$

Then the forward libor rate $r_t^{T,\widetilde{T}^{\mathtt{N}}}$ can be defined as:

$$\begin{aligned} r_t^{T,\widetilde{T}^{\mathtt{N}}} &= \frac{1}{\widetilde{T}^{\mathtt{N}} - T}\left(\frac{P_t^T}{P_t^{\widetilde{T}^{\mathtt{N}}}} - 1\right) = \frac{1}{\widetilde{T}^{\mathtt{N}} - T}\left(\mathfrak{P}_t - 1\right) \\ \Longrightarrow \mathfrak{P}_t &= 1 + \left(\widetilde{T}^{\mathtt{N}} - T\right) r_t^{T,\widetilde{T}^{\mathtt{N}}} \end{aligned}$$

We know the percentage vol of $\left(\widetilde{T}^{\mathtt{N}} - T\right) r_t^{T,\widetilde{T}^{\mathtt{N}}}$ is $\sigma^{r^{T,\widetilde{T}^{\mathtt{N}}}}$, i.e. the percentage volatility of the forward libor rate $r_t^{T,\widetilde{T}^{\mathtt{N}}}$. With some simple algebra† it turns out that the percentage volatility $\sigma^{\mathfrak{P}}$ of $\mathfrak{P}_t$ is given by:

$$\sigma^{\mathfrak{P}} = \frac{\left(\widetilde{T}^{\mathtt{N}} - T\right) r_t^{T,\widetilde{T}^{\mathtt{N}}}}{1 + \left(\widetilde{T}^{\mathtt{N}} - T\right) r_t^{T,\widetilde{T}^{\mathtt{N}}}}\sigma^{r^{T,\widetilde{T}^{\mathtt{N}}}} = \left(1 - \frac{P_t^{\widetilde{T}^{\mathtt{N}}}}{P_t^T}\right)\sigma^{r^{T,\widetilde{T}^{\mathtt{N}}}}$$

Case 2 $T = \widetilde{T}^{\mathtt{N}}$

Then obviously $\sigma^{\mathfrak{P}} = 0$.

Case 3 $T > \widetilde{T}^{\mathtt{N}}$

Then we write $\mathfrak{P}_t$ in terms of the forward libor rate $r_t^{\widetilde{T}^{\mathtt{N}},T}$:

$$\mathfrak{P}_t = \left[1 + \left(T - \widetilde{T}^{\mathtt{N}}\right) r_t^{\widetilde{T}^{\mathtt{N}},T}\right]^{-1}$$

The percentage volatility of $r_t^{\widetilde{T}^{\mathtt{N}},T}$ is $\sigma^{r^{\widetilde{T}^{\mathtt{N}}T}}$. With a bit of algebra‡ we find:

$$\sigma^{\mathfrak{P}} = -\frac{\left(T - \widetilde{T}^{\mathtt{N}}\right) r_t^{\widetilde{T}^{\mathtt{N}},T}}{1 + \left(T - \widetilde{T}^{\mathtt{N}}\right) r_t^{\widetilde{T}^{\mathtt{N}},T}}\sigma^{r^{\widetilde{T}^{\mathtt{N}},T}} = \left(1 - \frac{P_t^T}{P_t^{\widetilde{T}^{\mathtt{N}}}}\right)\sigma^{r^{\widetilde{T}^{\mathtt{N}},T}}$$

Recapitulating, we have:

$$\sigma^{\mathfrak{P}} = \begin{cases} \left(1 - \frac{P_t^{\widetilde{T}^{\mathtt{N}}}}{P_t^T}\right)\sigma^{r^{T,\widetilde{T}^{\mathtt{N}}}} & \text{if} \quad T \le \widetilde{T}^{\mathtt{N}} \\ \left(1 - \frac{P_t^T}{P_t^{\widetilde{T}^{\mathtt{N}}}}\right)\sigma^{r^{\widetilde{T}^{\mathtt{N}},T}} & \text{if} \quad \widetilde{T}^{\mathtt{N}} \le T \end{cases} \tag{3.27}$$

By combining (3.19),(3.22),(3.26),(3.27) we get for the drift of the forward swap rate in $\mathcal{Q}^P$:

†We use the fact that for a positive process F: $\frac{dF}{F} = \mu dt + \sigma dw \Rightarrow \frac{d(1+F)}{1+F} = (\ldots)dt + \frac{F}{1+F}\sigma dw$

‡We use the fact that for a positive process F: $\frac{dF}{F} = \mu dt + \sigma dw \Rightarrow \frac{d[(1+F)^{-1}]}{(1+F)^{-1}} = (\ldots)dt - \frac{F}{1+F}\sigma dw$

$$\mu^R = \sigma^R \left(\sigma^R + \sigma^{\mathfrak{P}} - \sigma^{r^{\widetilde{T}^0, \widetilde{T}^{\mathtt{N}}}} \right)$$

The equation above is exact but not very practical because both $\sigma^{\mathfrak{P}}$ and $\sigma^{r^{\widetilde{T}^0, \widetilde{T}^{\mathtt{N}}}}$ are generally non-deterministic stochastic processes. To proceed we need to make the following two simplifying assumptions:

Assumption 3.4 [$\boldsymbol{Lognormal\ Libor\ Rates\ Model}$] *The percentage volatilities* $\sigma^{r^{\widetilde{T}^0, \widetilde{T}^{\mathtt{N}}}}$ *and* $\sigma^{r^{\mathtt{MIN}(T, \widetilde{T}^{\mathtt{N}}), \mathtt{MAX}(T, \widetilde{T}^{\mathtt{N}})}}$ *of the forward libor rates* $r^{\widetilde{T}^0, \widetilde{T}^{\mathtt{N}}}$ *and* $r^{\mathtt{MIN}(T, \widetilde{T}^{\mathtt{N}}), \mathtt{MAX}(T, \widetilde{T}^{\mathtt{N}})}$ *respectively, are deterministic bounded functions of time.*

Assumption 3.5 [$\boldsymbol{Lognormal\ Bond\ Ratio\ Model}$] *The percentage volatility* $\sigma^{\mathfrak{P}}$ *of the bond ratio process* $\mathfrak{P}^{T, \widetilde{T}^{\mathtt{N}}}$ *is deterministic bounded function of time.*

One might think that we cannot have both swap rates and libor rates lognormally distributed. Note though that the libor rates involved in *assumption* 3.4 are not the same with those used in the definition of the swap rate. Therefore the *assumption* 3.4 does not necessarily violate the no arbitrage condition. Clearly though, *assumption* 3.5 is not compatible with *assumption* 3.4, due to the relation 3.27

In order to satisfy *assumption* 3.5, we could in some approximating sense replace the fraction of bond prices in 3.27 with its initial value. We do not need to change the libor rate volatilities since these are already deterministic (but possibly time dependent) from *assumption* 3.4:

$$\sigma^{\mathfrak{P}} \simeq \begin{cases} \left(1 - \frac{P_0^{\widetilde{T}^{\mathtt{N}}}}{P_0^T}\right) \sigma^{r^{T, \widetilde{T}^{\mathtt{N}}}} & \text{if} \quad T \leq \widetilde{T}^{\mathtt{N}} \\ \left(1 - \frac{P_0^T}{P_0^{\widetilde{T}^{\mathtt{N}}}}\right) \sigma^{r^{\widetilde{T}^{\mathtt{N}}, T}} & \text{if} \quad \widetilde{T}^{\mathtt{N}} \leq T \end{cases}$$

3.1 Special Case: European Swaption

Assume the special case when the fixing time $\widetilde{T}$, the start time $\widetilde{T}^0$ and the payment time T all coincide, i.e. $\widetilde{T} = \widetilde{T}^0 = T$. Let further the amount paid at $\widetilde{T}^0$ be given by $\left(\sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_{\widetilde{T}^0}^{\widetilde{T}^i}\right) \mathtt{MAX}\left(R_{\widetilde{T}^0} - K, 0\right)$ where K some constant. Note this is not of the form (3.1) and particularly is not linear in $R_{\widetilde{T}^0}$. Instead, it can be verified this is the payout at the expiry time $\widetilde{T}^0$ of a unit notional european swaption with strike at K, when the underlying swap is the one associated with the swap rate $R_{\widetilde{T}^0}$ being set at time $\widetilde{T}^0$. Interestingly enough, moving to measure $\mathcal{Q}^P$ and making use of *result* 3.3, is not helpful here because the payout cannot be written as a function of $R_{\widetilde{T}^0}$.

The way to value the european swaption is by choosing the positive process $\sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_t^{\widetilde{T}^i}$, $t \leq \widetilde{T}^0$ as numeraire. Let $\mathcal{Q}^R$ the associated equivalent martingale measure. Let V_t the value of this swaption at any earlier time $t \leq \widetilde{T}^0$. Then V_t divided by the numeraire is martingale w.r.t. $\mathcal{Q}^R$
$\Rightarrow V_0 = \sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_0^{\widetilde{T}^i} \mathsf{E}_0^{\mathcal{Q}^R} \left(\mathtt{MAX}\left(R_{\widetilde{T}^0} - K, 0\right) \right)$

The SDE of R_t in $\mathcal{Q}^R$ given in (3.11) leads to the immediate calculation of the expectation:

$$\mathsf{E}_0^{\mathcal{Q}^R} \left(\mathtt{MAX}\left(R_{\widetilde{T}^0} - K, 0\right) \right) = R_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2) \tag{3.28}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{R_0}{K}\right) + \frac{1}{2}\overline{\sigma^R}^2 \widetilde{T}^0}{\overline{\sigma^R}\sqrt{\widetilde{T}^0}} \\ d_2 &= d_1 - \overline{\sigma^R}\sqrt{\widetilde{T}^0} \\ \mathcal{N}() &= \text{Standard normal cumulative distribution function} \\ \overline{\sigma^R} &= \frac{\int_0^{\widetilde{T}^0} \sigma_u^R \, du}{\widetilde{T}^0} \end{aligned} \quad (3.29)$$

and R_0 stands for the forward swap rate $R_0^{\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}}$ as seen at time 0, i.e. today.

We get therefore the final result:

$$V_0 = \sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_0^{\widetilde{T}^i} \left[R_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2) \right] \quad (3.30)$$

The expression $\sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_0^{\widetilde{T}^i}$ is also called the “*spot dv01*” of the swap.

(3.30) can be solved for $\overline{\sigma^R}$ when the price V_0 is known from the market. In this context, the $\overline{\sigma^R}$ is called the “*implied Black swaption volatility*”. For the purpose of exotics pricing we would rather know σ_u^R for all times u — not just its time average up to some fixed time $\widetilde{T}^0$. We will see in *section* 3.3 that a european option with expiry $\widetilde{T}'$ on a forward swap starting at the later time $\widetilde{T}^0$, provides information on the time average volatility up to time $\widetilde{T}'$. If we knew the $\overline{\sigma^R}$ for all possible expiries $\widetilde{T}'$ such that $0 \le \widetilde{T}' \le \widetilde{T}^0$ for fixed $\widetilde{T}^0$, we could invert (3.29) and solve for σ_u^R , $0 \le u \le \widetilde{T}^0$, as a function of time. Alternatively, if we cannot find liquid options on forward swaps, we may simply use the volatility $\overline{\sigma^R}$ of the vanilla european swaption (i.e. option on a spot swap) expiring at $\widetilde{T}^0$ and assume σ_u^R is constant, i.e. $\sigma_u^R = \overline{\sigma^R}$ for all u such that $0 \le u \le \widetilde{T}^0$.

3.2 Special Case: Caplet

We assume the special case when the fixing time $\widetilde{T}$ and the start time $\widetilde{T}^0$ coincide. Let the rate $R_{\widetilde{T}^0}$ fixed at time $\widetilde{T}^0$, consisted from the single period $[\widetilde{T}^0, \widetilde{T}^1]$, $\widetilde{T}^0 < \widetilde{T}^1$, i.e. $\mathtt{N} = 1$. Then we are dealing with a spot (w.r.t. fixing time $\widetilde{T}^0$) libor rate, so let denote it by $r_{\widetilde{T}^0} \equiv r_{\widetilde{T}^0}^{\widetilde{T}^0,\widetilde{T}^1}$.

Let also the cash-flow $\tau \mathtt{MAX}\Big(r_{\widetilde{T}^0} - K, 0\Big)$ paid at $\widetilde{T}^1$ (i.e. we assume the payment time T and first coupon time $\widetilde{T}^1$ coincide as well). This is the payout of a unit notional caplet on $r_{\widetilde{T}^0}$ with strike at K. Since this can be written as a function of $r_{\widetilde{T}^0}$, we can value it easily in the $\mathcal{Q}^P$ measure by making use of the *result* 3.3. In this special case $\mathcal{Q}^R = \mathcal{Q}^P$, since the respective numeraires are the same — apart from a constant multiplier — $\Rightarrow$ the SDE (3.11) applies in $\mathcal{Q}^P$ as well. Now direct application of (4.8) gives:[†]

$$V_0 \equiv V_0\Big(\tau\mathtt{MAX}\Big(r_{\widetilde{T}^0} - K, 0\Big)\Big) = P_0^{\widetilde{T}^1}\left[\mathsf{E}_0^{\mathcal{Q}^P}\left(\tau\mathtt{MAX}\Big(r_{\widetilde{T}^0} - K, 0\Big)\right)\right] = P_0^{\widetilde{T}^1}\tau\left[\mathsf{E}_0^{\mathcal{Q}^P}\left(\mathtt{MAX}\Big(r_{\widetilde{T}^0} - K, 0\Big)\right)\right]$$

As discussed, the SDE (3.11) applies, which leads to the final result:

[†]Assuming the swaption formula (3.30) known, we could simply apply it here, since a caplet is nothing but a swaption with the underlying swap consisted of a single period. Nevertheless we provide here a derivation independent of the swaption result.

$$V_0 = P_0^{\widetilde{T}^1} \tau \left[r_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2) \right] \tag{3.31}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{r_0}{K}\right) + \frac{1}{2}\overline{\sigma^r}^2 \widetilde{T}^0}{\overline{\sigma^r}\sqrt{\widetilde{T}^0}} \\ d_2 &= d_1 - \overline{\sigma^r}\sqrt{\widetilde{T}^0} \\ \overline{\sigma^r} &= \frac{\int_0^{\widetilde{T}^0} \sigma_u^r du}{\widetilde{T}^0} \end{aligned} \tag{3.32}$$

and r_0 stands for the forward libor rate $r_0^{\widetilde{T}^0,\widetilde{T}^1}$ as seen at time 0, i.e. today.

(3.31) can be solved for $\overline{\sigma^r}$ when the price V_0 is known from the market. In this context, the $\overline{\sigma^r}$ is called the "*implied Black caplet volatility*". Like in the swaption case discussed above, for the purpose of exotics pricing we would rather know σ_u^r for all times u — not just its time average up to some fixed time $\widetilde{T}^0$. We will see in *section* 3.4 that a european option with expiry $\widetilde{T}'$ on a forward libor starting at the later time $\widetilde{T}^0$, provides information on the time average volatility up to time $\widetilde{T}'$. If we knew the $\overline{\sigma^r}$ for all possible expiries $\widetilde{T}'$ such that $0 \leq \widetilde{T}' \leq \widetilde{T}^0$ for fixed $\widetilde{T}^0$, we could invert (3.32) and solve for σ_u^r , $0 \leq u \leq \widetilde{T}^0$, as a function of time. Alternatively, if we cannot find liquid options on forward libor, we may simply use the volatility $\overline{\sigma^r}$ of the vanilla caplet (i.e. option on a spot libor) expiring at $\widetilde{T}^0$ and assume σ_u^r is constant, i.e. $\sigma_u^r = \overline{\sigma^r}$ for all u such that $0 \leq u \leq \widetilde{T}^0$.

3.3 Special Case: European Option on a Forward Swap

This is an extension of the european swaption of *section* 3.1. There the underlying at option expiry was a spot swap. Here the underlying is a forward swap. Mathematically we express this fact by allowing the fixing time $\widetilde{T}$ being earlier than the start time $\widetilde{T}^0$, i.e. $\widetilde{T} \leq \widetilde{T}^0$. In the limit when $\widetilde{T} = \widetilde{T}^0$, we recover the european swaption case.

Like in *section* 3.1, we assume the payout is received at fixing time, i.e. $T = \widetilde{T}$. One can verify the appropriate amount is $\left(\sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_{\widetilde{T}}^{\widetilde{T}^i}\right) \mathtt{MAX}\left(R_{\widetilde{T}} - K, 0\right)$, where $R_{\widetilde{T}}$ is the forward swap rate fixed at time $\widetilde{T}$, i.e. $R_{\widetilde{T}} \equiv R_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1,\dots,\widetilde{T}^{\mathtt{N}}}$.

We value this instrument by the same procedure followed in *section* 3.1. We choose the numeraire $\sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_t^{\widetilde{T}^i}$, $0 \leq t \leq \widetilde{T}$ with the associated measure $\mathcal{Q}^R$. Then the martingale property implies: $V_0 = \sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_0^{\widetilde{T}^i} \mathsf{E}_0^{\mathcal{Q}^R} \left(\mathtt{MAX}\left(R_{\widetilde{T}} - K, 0\right) \right)$

The SDE of R_t in $\mathcal{Q}^R$ given in (3.11) leads to the immediate calculation of the expectation:

$$\mathsf{E}_0^{\mathcal{Q}^R} \left(\mathtt{MAX}\left(R_{\widetilde{T}} - K, 0\right) \right) = R_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2) \tag{3.33}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{R_0}{K}\right) + \frac{1}{2}\overline{\sigma^R}^2 \widetilde{T}}{\overline{\sigma^R}\sqrt{\widetilde{T}}} \\ d_2 &= d_1 - \overline{\sigma^R}\sqrt{\widetilde{T}} \\ \overline{\sigma^R} &= \frac{\int_0^{\widetilde{T}} \sigma_u^R du}{\widetilde{T}} \end{aligned} \tag{3.34}$$

and R_0 stands for the forward swap rate $R_0^{\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}}$ as seen at time 0, i.e. today.

We get therefore the final result:

$$V_0 = \sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i P_0^{\widetilde{T}^i} \left[R_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2)\right] \tag{3.35}$$

This is actually the same formula with (3.30) except from $\widetilde{T}^0$ in the three equations (3.29) being replaced with $\widetilde{T}$ in the three equations (3.34). The practical result is that for a fixed underlying swap starting at $\widetilde{T}^0$, the option on this swap expiring at $\widetilde{T}^0$ — i.e. the vanilla swaption — requires knowledge of the time average volatility of σ_u^R up to time $\widetilde{T}^0$, whereas the option on the forward of this swap expiring at $\widetilde{T}$ requires knowledge of the time average volatility of σ_u^R up to time $\widetilde{T}$. This justifies the claim we made in *section* 3.1 regarding the recovering of the time dependent function σ_u^R from the option on forward swap market data.

3.4 Special Case: European Option on a Forward Libor

This can be considered as an extension of the vanilla caplet, where the underlying fixed at expiry is some forward libor rate instead of the spot libor rate. Like before we denote by $\widetilde{T}$ the expiry of the option, which is the same with the fixing time of the forward libor running from $\widetilde{T}^0$ to $\widetilde{T}^1$. The payout occurs at time $\widetilde{T}^1$ (i.e. $T = \widetilde{T}^1$) and concerns the amount $\tau\mathtt{MAX}\left(r_{\widetilde{T}} - K, 0\right)$. We may value this instrument like in *section* 3.2, by choosing as numeraire the bond $P^{\widetilde{T}^1}$ and applying the martingale relationship. Alternatively, we may observe this is a special case of the european option on a forward swap in the limit when the swap reduces to a single period. Then we can apply (3.35) directly to get:

$$V_0 = P_0^{\widetilde{T}^1} \tau \left[R_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2)\right] \tag{3.36}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{r_0}{K}\right) + \frac{1}{2}\overline{\sigma^r}^2 \widetilde{T}}{\overline{\sigma^r}\sqrt{\widetilde{T}}} \\ d_2 &= d_1 - \overline{\sigma^r}\sqrt{\widetilde{T}} \\ \overline{\sigma^r} &= \frac{\int_0^{\widetilde{T}} \sigma_u^r du}{\widetilde{T}} \end{aligned} \tag{3.37}$$

and r_0 stands for the forward libor rate $r_0^{\widetilde{T}^0,\widetilde{T}^1}$ as seen at time 0, i.e. today.

This is actually the same formula with (3.31) except from $\widetilde{T}^0$ in the three equations (3.32) being replaced with $\widetilde{T}$ in the three equations (3.37). The practical result is that for a fixed underlying libor starting at $\widetilde{T}^0$, the option on this libor expiring at $\widetilde{T}^0$ — i.e. the caplet — requires knowledge of the time average volatility of σ_u^r up to time $\widetilde{T}^0$, whereas the option on the forward of this libor expiring at $\widetilde{T}$ requires knowledge of the time average volatility of σ_u^r up to time $\widetilde{T}$. This justifies the claim we made in *section* 3.2 regarding the recovering of the time dependent function σ_u^r from the option on forward libor market data.

3.5 Special Case: Libor in Advance Floating Coupon

We consider here the valuation of a coupon on the floating leg of a unit notional vanilla swap, where the index is the spot† libor $r_{\widetilde{T}^0} \equiv r_{\widetilde{T}^0}^{\widetilde{T}^0,\widetilde{T}^1}\ ,\ 0 \leq \widetilde{T}^0 < \widetilde{T}^1$ where $(\widetilde{T}^0, \widetilde{T}^1)$ is the defining period of the libor rate. We also assume the payment follows exactly at the end of this period, i.e. $T = \widetilde{T}^1$. Note we do not impose any requirement on the coupon accrual period which determines τ. Also since the rate is just a libor rate, $\mathtt{N} = 1$. Finally the amount paid at $\widetilde{T}^1$ is $\tau r_{\widetilde{T}^0}$.

We value this coupon by using *result* 3.3 directly. Let V_0 its value today. Then (4.8) implies:

$$V_0 = P_0^{\widetilde{T}^1} \left[\mathsf{E}_0^{\mathcal{Q}^P} \left(\tau r_{\widetilde{T}^0} \right) \right] = P_0^{\widetilde{T}^1} \tau \left[\mathsf{E}_0^{\mathcal{Q}^P} \left(r_{\widetilde{T}^0} \right) \right]$$

but since $r_t\ ,\ 0 \leq t \leq \widetilde{T}^0$ is a martingale w.r.t. $\mathcal{Q}^P$:

$$V_0 = P_0^{\widetilde{T}^1} \tau r_0 \tag{3.38}$$

where of'course $r_0 \equiv r_0^{\widetilde{T}^0,\widetilde{T}^1}$ is the forward libor rate for the interval $(\widetilde{T}^0, \widetilde{T}^1)$ as seen from today.

(3.38) implies we can price this simple case of the vanilla coupon — where the index is the spot† libor and the coupon is paid at the end of the libor period — by replacing the floating amount $r_{\widetilde{T}^0}$ with the constant r_0 and then treat it as a fixed payment paid at $\widetilde{T}^1$, i.e. multiply it with $P_0^{\widetilde{T}^1}$. We will see in *section* 3.6 we can still apply the same prescription even when we replace the spot libor with the forward libor in the definition of the index, to the extent the coupon is still paid at the end of the libor period $\widetilde{T}^1$. “Unfortunately” this “logic” breaks down when the coupon is not paid exactly at the end of the libor period or when the index is not a forward on some libor rate. The latter case has been already treated in *result* 3.2, where we saw we may still apply the prescription mentioned here but with the precaution that the forward swap rate R_0 needs to be multiplied with the convexity correction factor $\mathfrak{C}^R$. The former case is treated in *section* .

3.6 Special Case: Forward Libor Floating Coupon

We consider here the valuation of a coupon on the floating leg of a unit notional swap, where the index is a forward libor $r_{\widetilde{T}} \equiv r_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1}\ ,\ 0 \leq \widetilde{T} \leq \widetilde{T}^0 < \widetilde{T}^1$ where $(\widetilde{T}^0, \widetilde{T}^1)$ is the defining period of the underlying libor rate. We also assume the payment follows exactly at the end of this period, i.e. $T = \widetilde{T}^1$. Note we do not impose any requirement on the coupon accrual period which determines τ. Since the rate is just a libor rate, $\mathtt{N} = 1$. The amount paid at $\widetilde{T}^1$ is $\tau r_{\widetilde{T}}$.

We value this coupon by using *result* 3.3 directly. Let V_0 its value today. Then (4.8) implies:

† “spot” with regard to the fixing time $\widetilde{T}^0$

$$V_0 = P_0^{\widetilde{T}^1}\left[\mathsf{E}_0^{\mathcal{Q}^P}\left(\tau r_{\widetilde{T}}\right)\right] = P_0^{\widetilde{T}^1}\tau\left[\mathsf{E}_0^{\mathcal{Q}^P}\left(r_{\widetilde{T}}\right)\right]$$

but since $r_t\,,\ 0 \le t \le \widetilde{T}$ is a martingale w.r.t. $\mathcal{Q}^P$:

$$V_0 = P_0^{\widetilde{T}^1}\tau r_0 \tag{3.39}$$

where of'course $r_0 \equiv r_0^{\widetilde{T}^0,\widetilde{T}^1}$ is the forward libor rate for the interval $(\widetilde{T}^0,\widetilde{T}^1)$ as seen from today.

(3.39) implies we can price this variation of the vanilla coupon — where the index is the forward libor and the coupon is paid at the end of the libor period — by replacing the floating amount $r_{\widetilde{T}}$ with the constant r_0 and then treat it as a fixed payment paid at $\widetilde{T}^1$, i.e. multiply it with $P_0^{\widetilde{T}^1}$. As we have already mentioned in the comment of *section* 3.5, this is clearly the same "prescription" with the one we used in that section for the vanilla case.

3.7 Special Case: Exotic Forward Libor Floating Coupon

This is an extension of the instrument treated in *section* 3.6. The attribute "exotic" refers to allowing the coupon payment time$T$ occur at any time on or after the fixing time $\widetilde{T}$. In *sections* 3.5 and 3.6 instead, the restriction $T = \widetilde{T}^1$ was imposed. The only restriction here — compared to the most general linear payout case treated in *result* 3.2 — is that the index consists of a single period, i.e. $\mathtt{N} = 1$.

So let $r_{\widetilde{T}} \equiv r_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1}$ be the forward libor rate fixed at time $\widetilde{T}$ and let $\tau r_{\widetilde{T}}$ be the amount that is paid at T with $0 \le \widetilde{T} \le \widetilde{T}^0 < \widetilde{T}^1$ and $\widetilde{T} \le T$. Let V_0 the value of this amount today. We may apply *result* 3.2 to get ($a = \tau$ and $b = 0$):

$$V_0 = P_0^T \tau r_0 \mathfrak{C}^r \tag{3.40}$$

where the convexity correction factor $\mathfrak{C}^r$ is given by:

$$\mathfrak{C}^r = e^{\int_0^{\widetilde{T}} \mu_u^r du}$$

and μ^r is given by (3.5), that is in our case:

$$\mu^r = \sigma^r\left(\sigma^r + \sigma^{\mathfrak{P}} - \sigma^{r^{\widetilde{T}^0,\widetilde{T}^1}}\right) = \sigma^r\sigma^{\mathfrak{P}}$$

since r stands for $r^{\widetilde{T}^0,\widetilde{T}^1}$ in the case we examine here.

So our final result for $\mathfrak{C}^r$ is:

$$\mathfrak{C}^r = e^{\int_0^{\widetilde{T}} \sigma^r \sigma^{\mathfrak{P}} du} \tag{3.41}$$

Note that $\sigma^{\mathfrak{P}}$ is approximated according to (3.6) by the expression:

$$\sigma^{\mathfrak{P}} \simeq \begin{cases} \left(1 - \frac{P_0^{\widetilde{T}^1}}{P_0^T}\right)\sigma^{r^{T,\widetilde{T}^1}} & \text{if} \quad T \le \widetilde{T}^1 \\ \left(1 - \frac{P_0^{T}}{P_0^{\widetilde{T}^1}}\right)\sigma^{r^{\widetilde{T}^1,T}} & \text{if} \quad \widetilde{T}^1 \le T \end{cases} \tag{3.42}$$

Note that $\sigma^{\mathfrak{P}} = 0$ for $T = \widetilde{T}^1$, which implies $\mathfrak{C}^r = 1$ in (3.41), as expected from our discussion in *section* 3.5.

(3.41) is only useful if we know the exact time dependence of σ^r and $\sigma^{r^{\mathtt{MIN}\left(T,\widetilde{T}^1\right),\mathtt{MAX}\left(T,\widetilde{T}^1\right)}}$. More often than not, we only know the time averages $\overline{\sigma^r}$ and $\overline{\sigma^{r^{\mathtt{MIN}(T,\widetilde{T}^1),\mathtt{MAX}(T,\widetilde{T}^1)}}}$ of these two quantities from the cap market. Then we could approximate (3.41) by:

$$\mathfrak{C}^r \simeq e^{\overline{\sigma^r}\,\overline{\sigma^{\mathfrak{P}}}\widetilde{T}} \tag{3.43}$$

where

$$\sigma^{\mathfrak{P}} \simeq \begin{cases} \left(1 - \frac{P_0^{\widetilde{T}^1}}{P_0^T}\right)\overline{\sigma^{r^{T,\widetilde{T}^1}}} & \text{if} \quad T \le \widetilde{T}^1 \\ \left(1 - \frac{P_0^{T}}{P_0^{\widetilde{T}^1}}\right)\overline{\sigma^{r^{\widetilde{T}^1,T}}} & \text{if} \quad \widetilde{T}^1 \le T \end{cases} \tag{3.44}$$

3.8 Special Case: Libor in Arrears Floating Coupon

Here the index is the spot libor rate $r_{\widetilde{T}^0} \equiv r_{\widetilde{T}^0}^{\widetilde{T}^0,\widetilde{T}^1}$ fixed at the same time when the coupon is paid, i.e. $\widetilde{T} = \widetilde{T}^0 = T$ and $0 \le \widetilde{T}^0 < \widetilde{T}^1$. This represents a subcase of the instrument studied in *section* 3.7 $\Rightarrow$ we may apply (3.40) to get:

$$V_0 = P_0^{\widetilde{T}^0}\tau r_0 \mathfrak{C}^r \tag{3.45}$$

where $r_0 \equiv r_0^{\widetilde{T}^0,\widetilde{T}^1}$ is the forward libor rate for the period $(\widetilde{T}^0, \widetilde{T}^1)$ as seen from today.

The convexity correction factor is approximated as $\mathfrak{C}^r \simeq e^{\overline{\sigma^r}\,\overline{\sigma^{\mathfrak{P}}}\widetilde{T}^0}$ from (3.43). Further on, $\sigma^{\mathfrak{P}}$ is given by (3.44) where we note that $T = \widetilde{T}^0$ and therefore the upper expression applies. We also observe that $\sigma^{r^{T,\widetilde{T}^1}} = \sigma^{r^{\widetilde{T}^0,\widetilde{T}^1}} = \sigma^r$. Therefore we have:

$$\sigma^{\mathfrak{P}} \simeq \left(1 - \frac{P_0^{\widetilde{T}^1}}{P_0^{\widetilde{T}^0}}\right)\overline{\sigma^r}$$

which leads to the final result:

$$\mathfrak{C}^r \simeq e^{\left(1-\frac{P_0^{\widetilde{T}^1}}{P_0^{\widetilde{T}^0}}\right)\left(\overline{\sigma^r}\right)^2\widetilde{T}^0} = e^{\frac{\widetilde{\tau} r_0}{1+\widetilde{\tau} r_0}\left(\overline{\sigma^r}\right)^2\widetilde{T}^0} \tag{3.46}$$

where in the last step we replaced $\frac{P_0^{\widetilde{T}^1}}{P_0^{\widetilde{T}^0}}$ by using $\frac{P_0^{\widetilde{T}^1}}{P_0^{\widetilde{T}^0}} = \frac{1}{1+\mathfrak{D}(\widetilde{T}^1-\widetilde{T}^0)r_0}$

Also $\widetilde{\tau} \equiv \mathfrak{D}\left(\widetilde{T}^1 - \widetilde{T}^0\right)$ is the day-count fraction of the interval $\widetilde{T}^0, \widetilde{T}^1$ according to the day-count convention used in the particular definition of the single compounded rate r.

As usually, $r_0 \equiv r_0^{\widetilde{T}^0,\widetilde{T}^1}$ is the forward libor rate for the period $(\widetilde{T}^0, \widetilde{T}^1)$ as seen from today.

The same approximating result for libor in arrears swaps (3.46) can be found in the finance literature, for example in [7, Page 126], where though, the derivation appears to be quite complex through multiple applications of Ito's lemma.

3.9 Special Case: Forward Libor in Arrears Floating Coupon

Here the index is the forward libor rate $r_{\widetilde{T}} \equiv r_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1}$ fixed at some time $\widetilde{T} \leq \widetilde{T}^0$. We demand that the libor period start at the same time when the coupon is paid, i.e. $\widetilde{T}^0 = T$ and $0 \leq \widetilde{T}^0 < \widetilde{T}^1$.

This represents an extension of the libor in arrears case studied in *section* 3.8. Basically the libor period $(\widetilde{T}^0, \widetilde{T}^1)$ stays the same, but the coupon is now based on the fixing of the forward libor rate at the earlier time $\widetilde{T}$. It's interesting to see what effect will this variation have in the convexity factor.

We note this still represents a subcase of the more general instrument studied in *section* 3.7 $\Rightarrow$ we may apply (3.40) to get:

$$V_0 = P_0^{\widetilde{T}^0} \tau r_0 \mathfrak{C}^r \tag{3.47}$$

where $r_0 \equiv r_0^{\widetilde{T}^0,\widetilde{T}^1}$ is the forward libor rate for the period $(\widetilde{T}^0, \widetilde{T}^1)$ as seen from today.

The convexity correction factor is approximated as $\mathfrak{C}^r \simeq e^{\overline{\sigma^r}\,\overline{\sigma^{\mathfrak{P}}}\widetilde{T}}$ from (3.43). Further on, $\sigma^{\mathfrak{P}}$ is given by (3.44) where we note that $T = \widetilde{T}^0$ and therefore the upper expression applies. We also observe that $\sigma^{r^{T,\widetilde{T}^1}} = \sigma^{r^{\widetilde{T}^0,\widetilde{T}^1}} = \sigma^r$. Therefore we have:

$$\sigma^{\mathfrak{P}} \simeq \left(1 - \frac{P_0^{\widetilde{T}^1}}{P_0^{\widetilde{T}^0}}\right) \overline{\sigma^r}$$

which leads to the final result:

$$\mathfrak{C}^r \simeq e^{\left(1-\frac{P_0^{\widetilde{T}^1}}{P_0^{\widetilde{T}^0}}\right)\left(\overline{\sigma^r}\right)^2 \widetilde{T}} = e^{\frac{\widetilde{\tau}\, r_0}{1+\widetilde{\tau}\, r_0}\left(\overline{\sigma^r}\right)^2 \widetilde{T}} \tag{3.48}$$

where in the last step we replaced $\frac{P_0^{\widetilde{T}^1}}{P_0^{\widetilde{T}^0}}$ by using $\frac{P_0^{\widetilde{T}^1}}{P_0^{\widetilde{T}^0}} = \frac{1}{1+\mathfrak{D}\left(\widetilde{T}^1-\widetilde{T}^0\right) r_0}$

Again, $r_0 \equiv r_0^{\widetilde{T}^0,\widetilde{T}^1}$ is the forward libor rate for the period $(\widetilde{T}^0, \widetilde{T}^1)$ as seen from today.

We observe that the only difference between (3.46) and (3.48) is the $\widetilde{T}^0$ in (3.46) is replaced by $\widetilde{T}$ in (3.48).

It's interesting to check the limit of (3.48) when we let $\widetilde{T} \to 0$. We get $\mathfrak{C}^r \to 1$ which is what we would expect, since in this case the index becomes deterministic from today's scope. In the limit where $\widetilde{T} \to \widetilde{T}^0$ we recover (3.46) as we should.

3.10 Special Case: Forward CMS Swap Floating Coupon with Cap and Floor

Here the index is some forward swap rate $R_{\widetilde{T}} \equiv R_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathrm{N}}}$ fixed at some time $\widetilde{T} \leq \widetilde{T}^0$ and restricted to lie in the interval $[\mathrm{F}, \mathrm{C}]$, where F and C are constants called "Floor" and "Cap" respectively. The coupon is paid at time T with the only constraint being $T \geq \widetilde{T}$. Formally, the index $I^{R_{\widetilde{T}}}$ in this case is not just $R_{\widetilde{T}}$ but rather is given by the expression $I^{R_{\widetilde{T}}} = \mathtt{MAX}\Big(\mathtt{MIN}\Big(R_{\widetilde{T}}, \mathrm{C}\Big), \mathrm{F}\Big)$. The amount paid out is $\tau I^{R_{\widetilde{T}}}$†.

† We assume for simplicity that the multiplier $\mathrm{m} = 1$ and the spread $\mathrm{s} = 0$. Otherwise the full coupon paid on a unit notional would be: $\tau(\mathrm{m} I^{R_{\widetilde{T}}} + \mathrm{s})$.

We may rewrite the index in the form $\mathrm{F} + \mathtt{MAX}\Big(R_{\widetilde{T}} - \mathrm{F}, 0\Big) - \mathtt{MAX}\Big(R_{\widetilde{T}} - \mathrm{C}, 0\Big)$, which implies we are actually dealing with an exotic caplet spread — portfolio of long, strike F, exotic caplet and a short, strike C, exotic caplet — plus a fixed payment of F. By "exotic caplet" we mean the caplet-like instrument where the index is a forward swap rate instead of a spot libor or where the index is a forward libor but the payment may occur at a time other than the end of the rate period. Let V_0 the value of this cash-flow today. We may calculate V_0 by applying *result* 3.3:

$$V_0 = P_0^T \left[\mathsf{E}_0^{\mathcal{Q}^P} \left(\tau I^{R_{\widetilde{T}}} \right) \right] = P_0^T \tau \left[\mathrm{F} + \mathsf{E}_0^{\mathcal{Q}^P} \left(\mathtt{MAX}\Big(R_{\widetilde{T}} - \mathrm{F}, 0\Big) \right) - \mathsf{E}_0^{\mathcal{Q}^P} \left(\mathtt{MAX}\Big(R_{\widetilde{T}} - \mathrm{C}, 0\Big) \right) \right]$$

We can easily evaluate the two expectations by making use of the SDE (3.4), which implies a lognormal distribution for the random variable $R_{\widetilde{T}}$. The result is:

$$\begin{aligned} \mathsf{E}_0^{\mathcal{Q}^P} \left(\mathtt{MAX}\Big(R_{\widetilde{T}} - \mathrm{F}, 0\Big) \right) &= R_0 e^{\overline{\mu^R}\widetilde{T}} \mathcal{N}\Big(d_1^{\mathrm{F}}\Big) - \mathrm{F}\mathcal{N}\Big(d_2^{\mathrm{F}}\Big) \\ \mathsf{E}_0^{\mathcal{Q}^P} \left(\mathtt{MAX}\Big(R_{\widetilde{T}} - \mathrm{C}, 0\Big) \right) &= R_0 e^{\overline{\mu^R}\widetilde{T}} \mathcal{N}\Big(d_1^{\mathrm{C}}\Big) - \mathrm{C}\mathcal{N}\Big(d_2^{\mathrm{C}}\Big) \end{aligned}$$

where (H below is any positive constant)

$$d_1^H = \frac{\ln\left(\frac{R_0}{H}\right) + \overline{\mu^R}\widetilde{T} + \frac{1}{2}\overline{\sigma^R}^2\widetilde{T}}{\overline{\sigma^R}\sqrt{\widetilde{T}}} \tag{3.49}$$

$$d_2^H = d_1^H - \overline{\sigma^R}\sqrt{\widetilde{T}} \tag{3.50}$$

$$\overline{\sigma^R} = \frac{\int_0^{\widetilde{T}} \sigma_u^R \, du}{\widetilde{T}} \tag{3.51}$$

$$\overline{\mu^R} = \frac{\int_0^{\widetilde{T}} \mu_u^R \, du}{\widetilde{T}} \tag{3.52}$$

As usually R_0 stands for the forward swap rate $R_0^{\widetilde{T}, \widetilde{T}^0, \ldots, \widetilde{T}^{\mathrm{N}}}$ as seen at time 0, i.e. today.

μ^R is given by (3.5) of *result* 3.1.

We present below the final result:

$$V_0 = P_0^T \tau \left\{ \mathrm{F} + R_0 e^{\overline{\mu^R}\widetilde{T}} \left[\mathcal{N}\Big(d_1^{\mathrm{F}}\Big) - \mathcal{N}\Big(d_1^{\mathrm{C}}\Big) \right] - \mathrm{F}\mathcal{N}\Big(d_2^{\mathrm{F}}\Big) + \mathrm{C}\mathcal{N}\Big(d_2^{\mathrm{C}}\Big) \right\} \tag{3.53}$$

where d_1^{F} , d_2^{F} , d_1^{C} , d_2^{C} are given by (3.49) and (3.50)
and $\overline{\mu^R}$ is given by (3.52).

Due to the importance and the wide applicability of the above result, we present below the corresponding expression when the coupon involves a multiplier m and a spread s as well. More precisely, $V_0^{m,s}$ below is today's price of the cash-flow $\tau(\mathrm{m}I^{R_{\widetilde{T}}} + \mathrm{s})$ at time T:

$$V_0^{m,s} = P_0^T \tau \mathrm{m} \left\{ \mathrm{F} + R_0 e^{\overline{\mu^R}\widetilde{T}} \left[\mathcal{N}\Big(d_1^{\mathrm{F}}\Big) - \mathcal{N}\Big(d_1^{\mathrm{C}}\Big) \right] - \mathrm{F}\mathcal{N}\Big(d_2^{\mathrm{F}}\Big) + \mathrm{C}\mathcal{N}\Big(d_2^{\mathrm{C}}\Big) \right\} + P_0^T \tau \mathrm{s} \tag{3.54}$$

The valuation formula (3.53) has the desired convergence behavior:

Namely, $\mathrm{F} \to \mathrm{C} \Rightarrow V_0 \to P_0^T \tau \mathrm{F}$, as expected since in this case the index will tend to equal F, i.e. in the limit the coupon payment at T will be the fixed amount $\tau\mathrm{F}$.

Also, if we remove the barriers, i.e. we let $\mathrm{F} \to 0$ and $\mathrm{C} \to \infty$, we get $V_0 \to P_0^T \tau R_0 e^{\overline{\mu^R}\widetilde{T}}$, which clearly agrees with the expression (3.8) of *result* 3.2, which applies on cash-flows linear in $R_{\widetilde{T}}$.

4 Quanto CMS: Foreign Index Paid in Domestic currency.

We turn now our attention to the more complex case where the index $I_{\widetilde{T}}$ is a forward swap rate but with respect to the risk free yield curve of some currency "RatCcy" — referred to as "Rate Currency" — possibly different from the currency "PmtCcy" — referred to as "Payment Currency" — in which the payment takes place. The time diagram of figure 1 applies here as it is. We will use the tilde symbol "$\widetilde{\ }$" to refer to quantities related with the rate currency economy. So we denote the rate currency forward swap rate as $\widetilde{R}_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^{\mathtt{N}}}$, or briefly — when the context is clear — as $\widetilde{R}_{\widetilde{T}}$. Like in *section* 3 we reserve a special notation for rate currency libor rates: $\widetilde{r}_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1}$ or briefly $\widetilde{r}_{\widetilde{T}}$. Note also the same notational discipline applies for the symbols $\widetilde{T},\ \widetilde{T}^0,\ \widetilde{T}^1,\ldots,\ \widetilde{T}^{\mathtt{N}}$, since they are also quantities related with the rate currency. In contrast we write T without a "$\widetilde{\ }$" when it comes to the payment time, since this is a quantity related to the payment currency.

The fundamentally new stochastic process of this section is the FX rate. Similarly to our handling of the interest rate, we will reserve the symbol S primarily for the *forward* FX rate, since the spot FX rate is nothing but a forward whose maturity equals the observation time. Formally we write $S_{\widehat{T}}^{\frac{\mathrm{PmtCcy}}{\mathrm{RatCcy}};\widehat{T}^0}$, where $\widehat{T}$ is the set time (or observation time), $\widehat{T}^0$ is the maturity of the forward and $\frac{\mathrm{PmtCcy}}{\mathrm{RatCcy}}$ defines the involved currencies†. In particular $S_{\widehat{T}}^{\frac{\mathrm{PmtCcy}}{\mathrm{RatCcy}};\widehat{T}}$ is the spot FX rate at time $\widehat{T}$ which equals the number of units of payment currency PmtCcy needed to buy 1 unit of rate currency RatCcy. In other words it represents the value at time $\widehat{T}$ of one unit RatCcy in terms of PmtCcy. In what follows we will assume the values of all assets are w.r.t. PmtCcy and we will write $S_{\widehat{T}}^{\widehat{T}^0}$ or briefly $S_{\widehat{T}}$ to mean the forward FX rate associated with the FX $\frac{\mathrm{PmtCcy}}{\mathrm{RatCcy}}$. We may think of the rate currency RatCcy as the "foreign" currency and of the payment currency PmtCcy as the "domestic" currency.

Replacing $\widetilde{R}_{\widetilde{T}}$ for the index I_i in (2.1) and (2.2) we get:

$$\mathrm{CP} = \mathrm{N}\left(\mathrm{m}\widetilde{R}_{\widetilde{T}} + \mathrm{s}\right)\tau \tag{4.1}$$

$$V_0(\mathrm{CP}) = \mathrm{N}\left(\mathrm{m}V_0\left(\widetilde{R}_{\widetilde{T}}\right) + P_0^T\mathrm{s}\right)\tau \tag{4.2}$$

where we dropped the coupon index i for simplicity.

P_0^T is the price today of a riskless bond w.r.t. PmtCcy with maturity T.

The challenge is to calculate $V_0\left(\widetilde{R}_{\widetilde{T}}\right)$.

Like in *section* 3, let $\mathcal{Q}^P$ be the equivalent martingale measure associated with the bond price $P_t^T,\quad 0 \le t \le T$ chosen as numeraire. Let $V_t \underset{def}{=} V_t\left(\widetilde{R}_{\widetilde{T}}\right)$. Then $\frac{V_t}{P_t^T}$ is a martingale w.r.t. $\mathcal{Q}^P$ and obviously $V_T = \widetilde{R}_{\widetilde{T}}$.

†We use a "$\widehat{\ }$" to refer to times related with the FX rate

Therefore the martingale property leads to the result: $\frac{V_0}{P_0^T} = \mathsf{E}_0^{\mathcal{Q}^P}\left(\frac{V_T}{P_T^T}\right) = \mathsf{E}_0^{\mathcal{Q}^P}\left(\frac{\widetilde{R}_{\widetilde{T}}}{1}\right) \Rightarrow$

$$V_0 = P_0^T \mathsf{E}_0^{\mathcal{Q}^P}\left(\widetilde{R}_{\widetilde{T}}\right) \tag{4.3}$$

Although the *equation* (4.3) only requires the evaluation of the expectation $\mathsf{E}_0^{\mathcal{Q}^P}\left(\widetilde{R}_{\widetilde{T}}\right)$, we would prefer to determine the SDE of $\widetilde{R}_t, \quad 0 \le t \le \widetilde{T}$ w.r.t. $\mathcal{Q}^P$ †.

Here is the result:

Result 4.1 [$\boldsymbol{SDE\ of\ Foreign\ Forward\ Swap\ Rate\ R_t\ in\ \mathcal{Q}^P}$]

$$\frac{1}{\widetilde{R}_t}\frac{d\widetilde{R}_t}{dt} = \left(\mu^{\widetilde{R}} - \rho^{\widetilde{R},S}\sigma^{\widetilde{R}}\sigma^S\right)dt + \sigma^{\widetilde{R}}dw \qquad \textit{w.r.t. } \mathcal{Q}^P \quad \textit{for} \quad 0 \le t \le \widetilde{T} \tag{4.4}$$

where
$\widetilde{R}_t$ *is a brief notation of* $\widetilde{R}_t^{\widetilde{T}^0,\widetilde{T}^1,\ldots,\widetilde{T}^N}$ *which is the foreign (i.e. w.r.t.* RatCcy *) forward swap rate as seen at time* t *when the underlying swap starts at time* $\widetilde{T}^0$ *and pays coupons at times* $\widetilde{T}^1,\ldots,\widetilde{T}^N$*.*
The order holds: $\widetilde{T} \le \widetilde{T}^0 < \widetilde{T}^1 < \ldots < \widetilde{T}^N$*.*
$\mu^{\widetilde{R}}$ *is the drift of* $\widetilde{R}_t$ *in the foreign economy w.r.t.* $\mathcal{Q}^{\widetilde{P}}$ *and is given by (3.5), where all quantities should be understood as the corresponding foreign ones. Here* $\mathcal{Q}^{\widetilde{P}}$ *is the equivalent martingale measure in the foreign economy w.r.t. the riskless foreign bond price* $\widetilde{P}_t^T$*.*
$\sigma^{\widetilde{R}}$ *is the percentage volatility of* $\widetilde{R}_t$*. It can be inferred from the swaption market in the foreign economy.*
$\mathcal{Q}^P$ *is the equivalent martingale measure associated with the numeraire being the domestic (i.e. w.r.t.* PmtCcy *) riskless bond price* P_t^T *maturing at the coupon payment time* T*.*

We may now apply *result* 4.1 to calculate the expectation in 4.3:

$$V_0 = P_0^T \widetilde{R}_0 e^{\int_0^{\widetilde{T}} \mu_u^{\widetilde{R}} du} e^{-\int_0^{\widetilde{T}} \rho_u^{\widetilde{R},S} \sigma_u^{\widetilde{R}} \sigma_u^S du} \tag{4.5}$$

This in turn can be set in 4.2 to get the price today of the whole coupon.

By comparing (4.5) with the well known result $V_0\left(r_{\widetilde{T}}\right) = P_0^T r_0$ in the special case when $r_{\widetilde{T}}$ is an amount paid at T and equal to the payment currency spot at $\widetilde{T}$ libor rate from $\widetilde{T}$ to T — i.e. when $r_{\widetilde{T}} \equiv R_{\widetilde{T}}^{\widetilde{T},T} \equiv r_{\widetilde{T}}^{\widetilde{T},T}$ —, and where by r_0 we mean the forward rate $r_0^{\widetilde{T},T}$, we derive the following result:

† See the relevant discussion in *section* 3.

Result 4.2 [$\boldsymbol{Quanto/CMS\ Convexity\ Correction\ for\ Cash\ Flows\ linear\ in\ \widetilde{R}_{\widetilde{T}}}$]

When the coupon payment $\mathbb{CP}$ *at time* T *is linear on the foreign forward swap rate* $\widetilde{R}_{\widetilde{T}} \equiv \widetilde{R}_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1,\ldots\widetilde{T}^{\mathtt{N}}}$, $\widetilde{T} \leq T$ *and* $\widetilde{T} \leq \widetilde{T}^0 < \widetilde{T}^1 < \ldots < \widetilde{T}^{\mathtt{N}}$*, i.e. if* $\mathbb{CP} = a\widetilde{R}_{\widetilde{T}} + b$*, with* a, b *constants, then its value today is:*

$$V_0(\mathbb{CP}) = P_0^T \left(a\widetilde{R}_0 \mathfrak{C}^{\widetilde{R}} \mathfrak{C}^{S;\widetilde{R}} + b \right) \tag{4.6}$$

Here $\mathfrak{C}^{\widetilde{R}}$ *is the same convexity correction factor given by (3.9) and is due to the index differing from the standard libor.*
$\mathfrak{C}^{S;\widetilde{R}}$ *is a new correction factor, typically called the "quanto correction factor", given by:*

$$\mathfrak{C}^{S;\widetilde{R}} = e^{-\int_0^{\widetilde{T}} \rho_u^{\widetilde{R},S} \sigma_u^{\widetilde{R}} \sigma_u^S du} \tag{4.7}$$

where $\rho^{\widetilde{R},S}$, $\sigma^{\widetilde{R}}$, σ^S *are as in result 4.1.*
We remind P_0^T *is today's price of the domestic — i.e. w.r.t.* PmtCcy *— riskless bond with maturity* T *and* $\widetilde{R}_0 \equiv \widetilde{R}_0^{\widetilde{T}^0,\widetilde{T}^1,\ldots\widetilde{T}^{\mathtt{N}}}$ *is the forward swap rate w.r.t.* RatCcy *as seen from today.*

Finally we may want to value some generic — possibly non-linear in $R_{\widetilde{T}}$ — cash-flow CF_T paid out at time T. The following result holds:

Result 4.3 [$\boldsymbol{Valuation\ Expression\ for\ Cash\ Flows\ which\ are\ functions\ of\ \widetilde{R}_{\widetilde{T}}}$]

Let CF_T *some cash-flow paid at time* T *in currency* PmtCcy*, which is a — possibly non-linear — function of* $\widetilde{R}_{\widetilde{T}} \equiv \widetilde{R}_{\widetilde{T}}^{\widetilde{T}^0,\widetilde{T}^1,\ldots\widetilde{T}^{\mathtt{N}}}$, $\widetilde{T} \leq T$ *and* $\widetilde{T} \leq \widetilde{T}^0 < \widetilde{T}^1 < \ldots < \widetilde{T}^{\mathtt{N}}$*, i.e.* $CF_T \underset{def}{=} f(\widetilde{R}_{\widetilde{T}})$*. Then its value today is given by:*

$$V_0(CF_T) = P_0^T \left[\mathsf{E}_0^{\mathcal{Q}^P} \left(f(\widetilde{R}_{\widetilde{T}}) \right) \right] \tag{4.8}$$

where P_t^T *is the price at* t *of the domestic — w.r.t.* PmtCcy *— riskless bond with maturity* T *and* $\mathcal{Q}^P$ *is the equivalent martingale measure associated with* P_t^T.
The expectation can be in principle calculated by making use of the SDE of $\widetilde{R}_t$, $t \leq \widetilde{T}$ *w.r.t.* $\mathcal{Q}^P$ *according to result 4.1.*

Proof of ***result*** **4.1**

Without repeating the argumentation used at the same stage of the proof of *result* 3.1, we will determine the SDE of $\widetilde{R}_t$ through its conditional expectations in $\mathcal{Q}^P$.

We may write $\widetilde{R}_t$ in terms of the foreign — i.e. w.r.t. RatCcy — riskless bond prices $\widetilde{P}$ as:

$$\widetilde{R}_t = \frac{\widetilde{P}_t^{\widetilde{T}^0} - \widetilde{P}_t^{\widetilde{T}^{\mathtt{N}}}}{\sum_{i=1}^{\mathtt{N}} \widetilde{\tau}_i \widetilde{P}_t^{\widetilde{T}^i}}$$

where $\widetilde{T}^i$, $i = 0, \ldots, \mathtt{N}$, are as shown in figure 1.
and $\widetilde{\tau}_i$ refers to the daycount fraction associated with the interval $(\widetilde{T}^{i-1}, \widetilde{T}^i)$, that is $\widetilde{\tau}_i \equiv \mathfrak{D}\left(\widetilde{T}^i - \widetilde{T}^{i-1}\right)$.

We consider now the measure $\mathcal{Q}^{\widetilde{P}}$, defined as the equivalent martingale measure in the RatCcy economy associated with the RatCcy-numeraire $\widetilde{P}_t^T$.

As we found in *section* 3, the rate $\widetilde{R}_t$ is diffused w.r.t. $\mathcal{Q}^{\widetilde{P}}$ according to (3.4), which we rewrite here with the new "$\sim$"-notation:

$$\frac{1}{\widetilde{R}_t}\frac{d\widetilde{R}_t}{dt} = \mu^{\widetilde{R}}dt + \sigma^{\widetilde{R}}dw \qquad \text{w.r.t. } \mathcal{Q}^{\widetilde{P}} \quad \text{for} \quad 0 \leq t \leq \widetilde{T} \tag{4.9}$$

An important fact for the rest of the proof, is that $\mathcal{Q}^{\widetilde{P}}$ is also an equivalent martingale measure w.r.t. the PmtCcy economy with $S_t^t\widetilde{P}_t^T$ being the corresponding numeraire. Note that S_t^t is the spot FX rate at t and therefore $S_t^t\widetilde{P}_t^T$ is the value of the RatCcy-bond in PmtCcy terms $\Rightarrow$ it represents the price of a PmtCcy-economy traded asset†. So we prove the following more general result:

Proposition 4.1 *Let economies $\mathbb{A}$ and $\mathbb{B}$ associated respectively with the class $\mathcal{A}$ and $\mathcal{B}$ of traded assets, when the economies do not interact . Assume next that the agents in economy $\mathbb{A}$ are allowed to trade at any time t on the assets of the economy $\mathbb{B}$ after they convert the $\mathbb{B}$-prices by multiplying them with the FX rate S_t. We assume this is also true for the agents of $\mathbb{B}$, with the only difference that they need to convert the $\mathbb{A}$-prices by dividing them with the FX rate S_t. This additional trading possibility will expand the class of $\mathbb{A}$-traded assets from $\mathcal{A}$ to $\mathcal{A}\cup S\mathcal{B}$, where $S\mathcal{B} = \{S_tB_t, \forall B_t \in \mathcal{B}\}$*

Assume next the existence of an equivalent martingale measure $\mathcal{Q}^{\mathbb{B}}$ in $\mathbb{B}$ associated with the numeraire $\mathcal{N}_t^{\mathbb{B}}$.

Then $\mathcal{Q}^{\mathbb{B}}$ is also an equivalent martingale measure in $\mathbb{A}$ but associated with the numeraire $S_t\mathcal{N}_t^{\mathbb{B}}$.

Proof

By the definition of "equivalent martingale measure" we have that the ratio $\frac{B_t}{\mathcal{N}_t^{\mathbb{B}}}$ is martingale w.r.t. $\mathcal{Q}^{\mathbb{B}}$, $\forall B \in \mathbb{B}$.

Now the assets available to the agents in $\mathbb{A}$ are represented by the class $\mathcal{A}\cup S\mathcal{B}$. It is enough to prove that both statements below hold:

1. $A_t \in \mathcal{A} \Rightarrow \frac{A_t}{S_t\mathcal{N}_t^{\mathbb{B}}}$ is a martingale w.r.t. $\mathcal{Q}^{\mathbb{B}}$.

2. $B_t \in \mathcal{B} \Rightarrow \frac{S_tB_t}{S_t\mathcal{N}_t^{\mathbb{B}}}$ is a martingale w.r.t. $\mathcal{Q}^{\mathbb{B}}$.

   The second statement is obvious due to the definition of $\mathcal{Q}^{\mathbb{B}}$ as the equivalent martingale measure in $\mathbb{B}$.

   For the first statement, observe that $\frac{A_t}{S_t}$ is a traded asset in the economy $\mathbb{B}$. Q.E.D.

†This follows from the definition of the "exchange rate" as a 1-dim diffusion S_t with the property: $A_t =$ asset price at t in RatCcy-economy $\Rightarrow S_tA_t =$ asset price at t in PmtCcy-economy.

References

[1] M. W. Baxter and A. J. O. Rennie, **Financial Calculus: An Introduction to Derivative Pricing**, *Cambridge University Press, Cambridge, UK, 1996*.

[2] I. V. Girsanov, **On Transforming a Certain Class of Stochastic Processes by Absolutely continuous Substitution of Measures**, *Theory of Probability and Applications, V5, 1960, pp 285 – 301*.

[3] J. M. Harrison and D. M. Kreps, **Martingales and Arbritrage in Multiperiod Securities Markets**, *Journal of Economic Theory, V20, 1979, pp 381 – 408*.

[4] J. M. Harrison and S. R. Pliska, **Martingales and Stochastic Integrals in the Theory of Continuous Trading**, *Stochastic Processes and their Applications, V11, 1981, pp 215 – 260*.

[5] R. S. Liptser and A. N. Shiryayev, **Statistics of Random Processes I: General Theory**, *Springer Verlag, New York, USA, 1977*.

[6] L. T. Nielsen, **Pricing and Hedging of Derivative Securities**, *Oxford University Press, New York, USA, 1999*.

[7] R. Rebonato, **Interest-Rate Option Models: Understanding, Analysing and Using Models for Exotic Interest-Rate Options**, *John Wiley & Sons Ltd, West Sussex, UK, 1996*.