Exercise Price Uncertainty, Risk Scaling Options and Payoff Allocations*

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ABSTRACT

We derive and present closed form solutions for risk scaling options (RSOs), a new class of stochastic exercise price options. We introduce choice parameters that allow contract counterparties to fix their expected risk exposure as desired. We show that RSOs generalize a wide set of marketed options including plain-vanilla options and can be customized to price options to buy or sell assets at premium or a discount to their market values. RSOs can be parameterized to yield negative Gammas and Vegas. We further demonstrate that any RSO with exercise price uncertainty can be represented by a portfolio of options with fixed exercise prices.

1 Introduction

Exercise price uncertainty arises naturally in off-exchange, non-traded options as well as in certain types of exchange-traded options. The relevant nature of the risk varies with respect to the nature of the underlying contract. In the case of non-traded or off-exchange options, even if the exercise price is fixed in terms of a chosen numeraire, defects in product quality and reliability, costly verification procedures, uncertain production costs, R&D costs, start-up costs, delivery lags and information asymmetries can all effectively lead to uncertain exercise prices. The control process is not as rigorous as in the case of exchange-traded options.

The literature is now rich with examples of options featuring stochastic exercise prices arising from path-dependence, such as asian options, lookback options, and barrier options. (See Zhang (1997) for a review of this literature.). Settlement prices in these contracts depend not only upon the price of the underlying asset at the time of exercise, but also upon historical prices of the underlying asset, realized during the life of the option.

In the earlier papers of Margrabe (1978) and Fisher (1978), exercise price uncertainty arises in a non-path-dependent manner. Margrabe solves the pricing problem of an option to exchange one asset for another, while Fisher extends the Black-Scholes model to the case in which the exercise price is a stochastic process following a geometric Brownian motion.

In this paper we initiate the study of a class of options that we term *risk-scaling options* (RSOs). RSOs exhibit exercise price uncertainty of a specific form, and represent one approach to risk modification. A RSO is a contingent claim with payoff function

$$[\pm(\beta S(t) - \lambda f(S(t)))]^{+}\mathbf{1}_{\{t\in\mathcal{T}\}}, \tag{1}$$

where $+$ and $-$ are associated with the call and put forms respectively, f is a continuous function and $\mathcal{T}$ is the set of admissible exercise times. In the sequel, we shall focus on the case in which

$$f(S) = \exp(\alpha \ln S + (1 - \alpha) \ln K) = K^{1-\alpha}S^{\alpha}$$

is the stated exercise price function. K, α, β and λ are assumed to be known constants at the time of purchase of the option and are agreed upon by the option writer and the buyer. RSOs belong both to the class of options with stochastic exercise prices as well as the class of nonlinear payoff options. Examples of options with effective uncertain exercise prices arise in various settings related to international finance, real estate financing, leasing and corporate finance.

As the real options literature has established, participation in a project with uncertain implementation costs, such as developing a new drug or the development of a mine, is analogous to purchasing a call option on the future cash flows generated by the project. The investment costs are analogous to the option premium in that paying the option premium gives the buyer claims to the residual cash flows associated with the venture. In the case of real options it may well be the case that there is uncertainty associated with the exercise price. As in the case of standard options, the option holder reserves the right not to exercise the option, can sell the option possibly to a third party or let the option expire.

It is natural in this environment to think of the parties involved in a non-traded contract wanting to reduce their risks. This leads to the consideration of risk-scaling, as parties to the contract try to achieve their risk reduction goals. Such mutual risk reduction can be achieved by the pooling of risks and/or agreeing on truncating cash flow streams to particular parties.

Examples of this type of behavior are seen in strategic alliances, joint ventures, venture capital financing and partnerships in which parties agree to share the payoffs associated with projects according to predetermined rules. Funding specified percentages of the uncertain development costs grants the investor the right to certain percentages of the projects' cash flows. The percentage of the development costs funded may differ from the percentage allocation of residual claims to the project's cash flows simply because the contacting parties have different bargaining strengths and/or different risk profiles.

For instance it is commonplace for venture capitalists to contribute say, 20% of the start-up costs for a project but control perhaps 50% of the equity of a company. Hence it is natural to think of American-style options with stochastic exercise prices, participating payoffs (in a sense to be described later) and uncertain maturity dates. Such options could be less costly (for the buyer) than non-participating options (depending on the scalar parametrization agreed on), could result in lower operating and financial leverage for the issuer and would permit true-risk sharing.[1]

The granting of such options should not be a costless exercise. The option grant transfers assets with tangible values to the option holder. From the standpoint of the granting agency/party an issue is how much should the option holder pay (the option premium) to receive the option. Examples of these non-traded options are grants of timber concessions, access to new R&D and participation in project development.

In all of these cases the party granting the option should want to know what value is being transferred. In addition the option recipient would want to know what are the potential payoffs associated with the option received. Typically an up-front fee is paid to participate in a project or a new development. We therefore need to be able to value options to purchase or sell assets (or a portion of an asset) when the exercise price is either at a premium to or at a discount from the asset price. Our study achieves this objective in the European case.

1.1 Literature Review and Our Extensions

[1] Even though American-style options are more common, their European counterparts arise frequently in international project finance. Partners in a project typically have fixed dates by which they must commit to projects and contribute their share of costs. There is no advantage to early exercise (paying their share of cost early generates no benefits as the payoff allocations are also typically fixed). Hence the early exercise premium is worthless and the real option in this case is properly valued as a European option.

As examples of financial options with stochastic exercise prices, we mention the European option to exchange one asset for another of Margrabe (1978) and the European option with an uncertain exercise price of Fischer (1978). Stulz (1982), Johnson (1987), Geske (1979) and Galai (1983) all treat models where exercise price uncertainty is involved because of optional arrangements on underlying assets. Stochastic exercise prices are a central feature in the case of path-dependent or "look-back" options such as Asian and barrier options.

The cross-currency option model analyzed by Rumsey (1991) generates an effective stochastic exercise price. In that model an option on a foreign currency asset is valued in dollars but has a fixed exercise price in terms of a third currency. In this case the uncertainty of the exercise price is generated by uncertainty with respect to the cross-rate between the dollar and the third currency at the time of exercise of the option. Similarly, Carr (1988) shows that compound call options will have stochastic exercise prices if their strike prices are denominated in a foreign currency.

Fischer (1978) develops a closed form solution in a model of exercise price uncertainty in which uncertainty enters the model in several forms. Merton (1973) also analyzes the case for American warrants with exercise prices that vary as a function of the length of time until expiration and shows that such a warrant is equivalent to a modified European warrant. The modification being that the owner is allowed to exercise the warrant at discrete times just prior to an exercise price change. Gu (2002) studies a class of options in which exercise prices are defined as a proportional discount to the price of the underlying asset.

We extend the current literature in several directions. The payoff function is generalized in such a way to accommodate an exercise price that is a combination of a deterministic function and a random function. Variation in the scaling and functional parameters allows for the derivation of the standard call option payoff function. We then derive two closed form solutions for a European call option with the generalized boundary condition. These solutions are equivalent.

One approach uses a decomposition method to express the payoff function in a more elementary form and uncovers a standard and a non-standard option. The pricing solution is the sum of these component values. The second solution approach does not involve any decomposition and solves for the option value of a non-standard option. What is new is that we show by virtue of the equivalence of the two solutions that a problem of exercise price uncertainty can be transformed into an equivalent problem involving a portfolio of fixed exercise price options. [2]

The parties to this non-standard option contract agree to the values of these parameters at the outset. This flexibility in the exercise function permits an easy extension to the payoff functions of more complex options that have appeared in the foreign exchange and capital markets. As such the paper

[2]This transformation is not unique as we have uncovered other transformations that lead to the solutions of other classes of option pricing problems. We will pursue these and other extensions in future papers.

has intimate links with that of Grabbe (1987), Boyle and Turnbull (1989) and Briys and Crouhy (1988) who treat the valuation of currency options with disappearing deductibles and proportional coverage, capped options etc.[3]

The second extension shows that the strike price being an average of two variables leads naturally to a comparison with the averaging-price options literature.[4] This literature is represented by the paper of Ritchken et al. (1990) and Kemna and Vorst (1990). However their models feature option pricing problems with deterministic exercise prices, that do not permit a closed-form solution because the average (arithmetic) price of the traded asset has a distribution that is generally unknown. Hence the Cox-Ross risk-neutrality argument could not be applied and they resort to Monte Carlo methods to derive analytical features of a numerical solution.

The model analyzed here does not encounter this difficulty. We derive closed-form solutions for the European version of the put and call. Our model prices options where the strike could be either at a premium or a discount to the current asset price. This is a natural extension of Gu(2002). Finally, we derive and present the Greeks for the model and show some new behavior for gammas and vegas.

[3]The put/call parity relation for foreign exchange options derived by Grabbe (1983) can easily be utilized to derive closed form solutions for the call variants of the hybrid options that they value. Hence the results derived in this paper can be shown to generate variants of the Briys and Crouhy (1988) results in a purely domestic setting.

[4]Options whose payouts are determined on the basis of the average of underlying asset values over some specified period are called Asian options. This class includes options whose strike prices are averages of past prices and whose effective asset price at contract maturity is some average of past prices. Unlike that class of options the strike price in our model is an average of two known prices, K and the underlying stock price..

The standard Black-Scholes framework in which the payoff functions are linear in the asset price $S(T)$ is a special case of the RSO environment (namely for $\alpha = 0, \lambda = \beta = 1$). In general, RSO payoff functions are non-linear. The degree of convexity is controlled by the parameters α, β and λ. So our pricing solution addresses more fundamentally a class of options with non-linear payoff functions.

This paper is organized as follows. In Section I the motivation for this study is described and the links with the existing literature are sketched. This allows us to cast additional light upon the general issue of option pricing with exercise price uncertainty. In Section II the model assumptions and specification are detailed and closed form solutions to the option pricing problem are presented. The closed form solutions for the put and the call option are then utilized to derive a put/call parity relation for this type of option problem.

Section III explores the properties such a solution must have and verifies that the solution obeys the standard Black-Scholes results for the case of $\alpha = 0$. In addition the solution parameters are varied in this generalized setting to demonstrate links with option pricing models in which the payoff function is scaled. It is demonstrated that the put/call parity satisfies the results previously presented by several authors.

Section IV analyzes how RSOs differ from standard options and explores their payoff scaling features. Section V analyzes its comparative static qualities “the Greeks”. We show that all vestiges of symmetry (between put and call) disappear when there is exercise price uncertainty. We show the convergence to standard Black-Scholes results when $\alpha = 0$, $\lambda = \beta = 1$and show why symmetry holds in some cases. Section VI concludes with some summary remarks.

2 The Model

We define a *market environment* consisting of a risky asset S and a risk-free bond S_0. The asset S can be considered to be the price of a stock or the value of a project. The asset may be traded or non-traded. The prices of these two assets are assumed to be governed by the stochastic differential equation,

$$dS(t) = (\mu - \delta)S(t)\,dt + \sigma S(t)dW(t), \quad t \in [0, T], \tag{2}$$

where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}$, W is a standard Brownian motion under a probability measure P, and

$$dS_0(t) = r\,dt, \quad t \in [0, T], \tag{3}$$

where $r \in \mathbb{R}^+$.

As is well-known, there are no arbitrage opportunities in such a market environment if and only if there exists a probability measure P_0 under which

S/S_0 is a martingale. Under this new measure, equations (2) and (3) become

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW_0(t), \quad t \in [0,T], \tag{4}$$

where, for all $t \in [0,T]$

$$W_0(t) := W(t) + \int_0^t \frac{\mu - \delta - r}{\sigma}\,du$$

is a standard Brownian motion under P_0, and

$$dS_0(t) = r\,dt, \quad t \in [0,T]. \tag{5}$$

A frictionless environment is posited where there are no penalties to short sales if the asset is a stock and there are no transactions costs attendant upon purchase or sale of the underlying asset or the option. The asset has a payout rate of δ.

3 Valuing European Risk-Scaling Options

A European risk-scaling call option is a contingent claim with payoff function

$$\phi_c(S(t);\rho) = (\beta S(t) - f(S(t)))^{+}\mathbf{1}_{\{t=T\}}, \tag{6}$$

where

$$f(S) = \lambda \exp(\alpha \ln S + (1-\alpha) \ln K) = \lambda K^{1-\alpha} S^{\alpha}$$

is the stated exercise price function. K, α, $\beta, \sigma, r, \tau, S_t$ and λ are assumed to be known at the time of purchase of the option. We assume that each counterparty has some information set and characteristics that are embedded in ϕ_i.The parameters α, β and λ are agreed upon by the option writer and the buyer, as the outcome of some bargaining process. In general, it is clear that α, β and λ are functions of K, σ, r, τ, S_t, ϕ_1 and ϕ_2, where ϕ_i subsumes all the information and characteristics of counterparty i.[5]

In this paper the process by which values of the parameters are determined is not developed. We impose the following rational parameter restrictions to ensure boundedness of the problem, $0 \leq \alpha \leq 1, 0 \leq \beta < \infty$ and $0 \leq \lambda < \infty$. [6]At expiration, the value of the call is

$$C(T, \rho; T) = (\beta S(T) - f(S(T)))^{+} \mathbf{1}_{\{t=T\}}, \tag{7}$$

The boundary condition for the put version of our option is

$$P(t, \rho; T) = (f(S(t)) - \beta S(t))^{+} \mathbf{1}_{\{t=T\}}, \tag{8}$$

We do not know a priori what the value of $S(T)$ is but its expected value (given time-t information) is $S(t)e^{(r)(T-t)}$ based on the assumed dynamics

[5]Hence the precise form of the parameters will vary depending on the case being analyzed.

[6]f is a Cobb-Douglas type function that has well known properties and is homogeneous of degree one in it K and S. α is a measure of the share of each term in the stated exercise price at the time the option is bought/sold. As S is stochastic the actual exercise price varies over time and will typically differ from the exercise price at the outset.

of the risk-neutralized asset process. Exercise price uncertainty enters the model through the value of the asset upon exercise of the option. Essentially this means that as the asset becomes more valuable, the strike price increases and is tied to the increased perceived value of the asset to be acquired. The parameter α is an index of exercise price uncertainty. It also determines uniquely the class of pricing problem for any arbitrary option with exercise price fitting our generalization.

As α tends to zero all exercise price uncertainty is resolved. The exercise price is λK. As α tends to unity the exercise price is λS. In the first case α tending to zero puts us in the standard Black-Scholes world. In the second case, paradoxically, even though the exercise price is completely random we will know from the outset, whether the option is valuable and will be exercised at maturity. For cases where $\beta > \lambda$ and $\alpha = 1$, the option will always be exercised at maturity if the company is alive. That is if its asset price is non-zero. Hence α resolves different types of uncertainty at its two limit points.

Intermediate values of α generate an averaging function of K and S. When $\alpha = .5$, the average is a geometric average. True exercise price depends on asset value, agreed upon participation rate and the investment costs if there were no uncertainty. β and λ are participation parameters that define the payoffs to the option buyer given the exercise price and the asset price. These latter measures define the proportionality of the payoffs. They can be specialized to situations where assets are purchased at a premium $\lambda > 1$, or at a discount $\lambda < 1$. In the case of beta, we can vary the value of the parameter to synthesize results typically associated with power options for example. We can associate values of $\beta > 1$ with power option type reward allocations.

The boundary conditions of a RSO call option are variants of those of the standard European call option. Indeed setting $\beta = 1, \lambda = 1$ and $\alpha = 0$ generates the usual boundary conditions. Setting $\lambda = \beta \neq 1$ and $\alpha = 0$ generates a hybrid call option that is the analog of the put option analyzed by Briys and Crouhy (1988)[7]. The recent proportional discount model of Gu (2002) is also just a special case of the American version of our model with $\alpha = 1, \beta = 1, \lambda < \beta$. He however does not discuss a solution for the European version of the model.

In this model the values of the parameters β, λ and α are not choice variables at contract maturity. Hence the holder of the option does not have a choice in terms of choosing between an index based on a truncated function of the stock price or a truncated function of a deterministic index as in the case of chooser options. However, the buyer potentially has a choice from among three alternatives at the time of purchase. For all values of α except 0 and 1, the exercise price will be based on an averaging function.

[7] They analyze a put with proportional coverage in which the option holder has partial protection against downward movement of the asset price below the exercise price. The hybrid call outlined here would only permit partial participation of the benefits in any upward movement of the asset price above the effective strike price. Hence it would be the direct analog of their results when applied to asset markets instead of currency markets. The novelty of our model is the option seller is only partially exposed to adverse consequences if the asset prices preform much better than anticipated.

3.1 European RSO

We present two solutions to the problem of RSO valuation. These solutions give different insights into the valuation problem. The first solution presented can be derived by using the general theory of contingent claims. The second solution decomposes the payoff function (with stochastic strike) into two functions with deterministic strikes. It transforms the problem from one with exercise price uncertainty into one with fixed strikes.

3.2 On the class of power options embedded in RSOs

The RSO class of options embeds multiple types of (traded) option contracts. This insight is clear from the decomposition approach. Using that approach we were able to derive and present solutions for a particular form of fractional power options. Esser (2003) presents solutions for power options in the case where $\alpha > 1$. Her solution does not explicitly address the problem for $\alpha < 1$, but solves problems in that class also. The solutions for our power options satisfy the generalized put/call parity relation implied by RSOs and Esser's results.

3.3 Pricing European Risk-Scaling Options

Theorem 3.1. *The value of the European RSO call option with parameters* $\rho = (\alpha, \beta, \lambda, K)$, *and expiration time* T *is*

$$C(t, S, \rho; T) = \beta\, e^{-\delta(T-t)} S N(d_1) - \lambda e^{-\alpha\delta(T-t)}\, \Upsilon(\rho)\, N(d_2) \tag{2}$$

where

$$d_1 := \frac{(1-\alpha)\ln(S/K) - \ln\lambda + \ln\beta + (1-\alpha)(r - \delta + \sigma^2/2)(T-t)}{(1-\alpha)\sigma\sqrt{(T-t)}} \tag{3}$$

$$d_2 := \frac{(1-\alpha)\ln(S/K) - \ln\lambda + \ln\beta + (1-\alpha)(r - \delta + \alpha\sigma^2 - \sigma^2/2)(T-t)}{(1-\alpha)\sigma\sqrt{(T-t)}} \tag{4}$$

$$\Upsilon(\rho) := K^{1-\alpha} S^{\alpha} \exp\{(\alpha - 1)(r + \alpha(\sigma^2/2))(T-t)\}$$

$N(\cdot)$ *denotes the cumulative normal density function.*

Proof. >From the theory of contingent claim valuation, it is straightforward to show that the value of the European RSC satisfies

$$C(t, S, \rho; T) = e^{-r(T-t)} E_0[\phi_c(S(T))|\mathcal{F}(t)] = \begin{cases} u(T-t, S(t)), & t < T \\ \phi_c(S(T)), & t = T, \end{cases}$$

where

$$u(\tau,x) = e^{-r\tau}\int_{\mathbb{R}} \phi_c(x\exp((r-\delta-\frac{1}{2}\sigma^2)\tau+\sigma z)\frac{1}{\sqrt{2\pi\tau)}}e^{-\frac{z^2}{2\tau}}\,dz.$$

Since $\phi_c(x\exp((r-\delta-\frac{1}{2}\sigma^2)\tau+\sigma z)) > 0$ if and only if

$$z > z^* := \frac{1}{\sigma}\left[\frac{1}{1-\alpha}\ln\left(\frac{\lambda K^{1-\alpha}x^\alpha}{\beta x}\right) - (r-\delta-\sigma^2/2)\tau\right],$$

we have that

$$\begin{aligned}
&e^{-r\tau}\int_{\mathbb{R}} \phi_c(x\exp((r-\delta-\frac{1}{2}\sigma^2)\tau+\sigma z))\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{z^2}{2\tau}}\,dz \\
&= e^{-r\tau}\int_{z^*}^{\infty}\{\beta x\exp((r-\delta-\frac{1}{2}\sigma^2)\tau+\sigma z) \\
&\qquad -\lambda K^{1-\alpha}x^\alpha\exp(\alpha(r-\delta-\frac{1}{2}\sigma^2)\tau+\sigma\alpha z)\}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{z^2}{2\tau}}\,dz.
\end{aligned}$$

Now, we split this last integral into a sum of two separate integrals I_1 and I_2 where

$$I_1 := e^{-r\tau}\int_{z^*}^{\infty}\beta x\exp((r-\delta-\frac{1}{2}\sigma^2)\tau+\sigma z)\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{z^2}{2\tau}}\,dz$$

and

$$I_2 := -e^{-r\tau}\int_{z^*}^{\infty}\lambda K^{1-\alpha}x^\alpha\exp(\alpha(r-\delta-\frac{1}{2}\sigma^2)\tau+\sigma\alpha z)\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{z^2}{2\tau}}\,dz.$$

Using the change of variable $z = -y\sqrt{s}+\sigma s$ in I_1 and $z = y\sqrt{s}+\alpha\sigma s$ in I_2, the result follows. $\square$

Analogously to the case for standard options, hedging is achieved by holding a portfolio consisting of positions in the underlying asset and the risk-free asset. Specifically, to remain perfectly hedged, the option writer must maintain a position in the underlying asset depending on the option's delta.

Standard put-call parity arguments can be used to price the European risk-scaling put option. The intuition here is as follows. Consider two portfolios, Portfolio A consists of the put option P. Portfolio B, that is long the call option C, short $\beta e^{-\delta(T-t)}$ units of the stock S, and holds a long position that is $\lambda e^{-\alpha\delta(T-t)}$ units of the "mutual fund" Υ. Under all possible scenarios neither portfolio is a dominated portfolio and hence by arbitrage reasoning they should be equivalently priced.

Corollary 3.1. *The value of the European RSO put is given by*

$$\begin{aligned} P(t,S,\rho;T) &= \lambda e^{-\alpha\delta(T-t)}\Upsilon(1-N(d_2)) - \beta e^{-\delta(T-t)}S(t)(1-N(d_1)) \\ &= C(t,S,\rho;T) - \beta e^{-\delta(T-t)}S(t,S,\rho;T) + \lambda e^{-\alpha\delta(T-t)}\Upsilon \end{aligned}$$

Corollary 3.2. *Generalized Put/Call Parity for RSOs must satisfy*

$$P(t,S,\rho;T) = C(t,S,\rho;T) - \beta e^{-\delta(T-t)}S(t,S,\rho;T) + \lambda e^{-\alpha\delta(T-t)}\Upsilon$$

It is a simple exercise to verify that when $\lambda = \beta = 1$, $\alpha = 0$, and

$\delta = 0$ that the usual put/call parity for non-dividend paying stocks $P = C - S + K\, e^{-r(T-t)}$ is derived.

By simply varying the intensity of α, β and λ parameters one is able to generate a wide variety of pricing results. For both the call and put formulas presented it is easily verifiable that when $\alpha = 0, \delta = 0,\ \lambda = \beta = 1$ that the standard Black-Scholes relations are generated. The model also satisfies one's intuition on option valuation when the exercise price is the price of the underlying asset. An option on an asset for which the exercise price is the asset price at contract maturity is worthless (providing $\lambda = \beta = 1$) and the model confirms this.

3.4 Decomposition Approach

The methods used to derive pricing results in the previous section are, of course, not comprehensive. It is intuitively appealing to consider the RSO call option as a portfolio of two options with deterministic exercise prices. Specifically, we note that the payoff function can be expressed (non-uniquely) as the difference of the payoff functions for an ordinary call option and that of a fractional power call option. Indeed, it is straightforward to see that we may write

$$\phi_c(t, S(t), \rho; T) = (\beta S(t) - \lambda K^{1-\alpha} S^{\alpha})^{+} \mathbf{1}_{\{t=T\}} \tag{9}$$

$$= \beta(S - \left(\frac{\lambda}{\beta}\right)^{\frac{1}{1-\alpha}} K)^{+} \mathbf{1}_{\{t=T\}} \tag{10}$$

$$-\lambda K^{(1-\alpha)}(S^{\alpha} - \left(\frac{\lambda}{\beta}\right)^{\frac{\alpha}{1-\alpha}} K^{\alpha})^{+} \mathbf{1}_{\{t=T\}}.$$

The European RSO call option is therefore equivalent to a portfolio of options consisting of a long position with β units of a standard call option on S with a strike price of $(\lambda/\beta)^{\frac{1}{1-\alpha}} K$ and a short position of $\lambda K^{(1-\alpha)}$ units of a fractional power option on S with a strike price of $(\lambda/\beta)^{\frac{\alpha}{1-\alpha}} K^{\alpha}$. Using this insight, we establish the following results.

Proposition 3.1. *The value of the European risk scaling call option with parameters $\rho = (\alpha, \beta, \lambda, K)$, and expiration time T is*

$$C(t, S, \rho; T) = \beta L_1(t, \rho; T) - \lambda K^{(1-\alpha)} L_2(t, \rho; T)$$

where

$$L_1 = \{S N(g_1) - K(\lambda/\beta)^{\frac{1}{1-\alpha}} \exp\{-r(T-t)\} N(g_2)\}$$

$$L_2 = \{(S^{\alpha} \exp\{(\alpha - 1)(r + (\alpha\sigma^2)/2)(T - t)\}) N(e_1) - ((\lambda/\beta)^{\frac{\alpha}{1-\alpha}} K^{\alpha} \exp\{-r(T-t)\}) N(e_2)\}$$

$$g_1 = \frac{\ln\,(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$g_2 = g_1 - \sigma\sqrt{T-t}$$

$$e_1 = \frac{(1-\alpha)\ln(S/K) - \ln\lambda + \ln\beta + (1-\alpha)(r + \alpha\sigma^2 - \sigma^2/2)(T-t)}{(1-\alpha)\sigma\sqrt{T-t}}$$

$$e_2 = e_1 - \alpha\sigma\sqrt{T-t}$$

and $N(\cdot)$ *is the cumulative normal density function.*

Proof. This result is a direct consequence of the decomposition in (9). Note that L_1 is the Black-Scholes price for a European call with exercise price $(\lambda/\beta)^{\frac{1}{1-\alpha}}K$. That L_2 is indeed the arbitrage price for the fractional power option follows from the results of Esser (2003). □

Esser (2003) also provides solutions for power and powered options with payoffs of $\max(S^a - K, 0)$ and $\max(S - K, 0)^a$. In both of these cases the power a is an arbitrary integer. Her solution generalizes the Black-Scholes solution also and is equal to it when $a = 1$. An added check on the accuracy of our results is that the pricing results using the decomposition approach also satisfy RSO put call parity.

We therefore show that a possibly non-traded, stochastic exercise price option can be replicated not only by holding positions in a traded asset and bonds but also by a synthetic position long in units of standard Black-Scholes types options (with a different strike price) and a short position in fractional power options. This result is also new in the options literature.

3.5 RSOs and Risk Sharing

The solution that is presented here has the analytically attractive feature of being applicable to a wide variety of options with scaled payoffs. Options with scaled patterns are naturally self-insuring to the option writer and buyer in the sense that they both can modify the basic payoff patterns to more closely conform to the requirements of their portfolio positions. However, from a fundamental viewpoint these types of options make it possible to alter the basic asymmetry in returns by limiting the maximum level of loss of the option writer. Such options are therefore intrinsically lower in value than standard options and the option premium should be correspondingly lower.

3.6 How RSOs Differ From Standard Options

With RSOs, contract participants can choose to share risk through the setting of the α, β and λ parameters. In the case of exercise price uncertainty α measures the percentage change in the exercise price relative to the increase in the underlying asset price. Hence let us say in project development the parties agree that increases in perceived project value should drive the costs of buying into the project, then as $\alpha \mapsto 1$,the value of any call option approaches zero except if $\beta \gg \lambda$. Hence in the limit as $\alpha \mapsto 1$, there is no sharing of the cost risks (except through the participation parameter λ). No one will purchase such an option (agree to such development terms) unless as mentioned $\beta \gg \lambda$ is accepted as a compensation factor to induce participation.

The other two parameters emphasize sharing through the payoff function. In the standard Black-Scholes model there is no sharing. The contract holder benefits in all the upside, above the contract exercise price. In a RSO contract we can have different forms of sharing on the cost side as well as on the benefits side. For instance if $\lambda = .5$, is the participation parameter for any development project, it means that the option granter has to participate equally in the project's costs for the project to be implemented.

RSOs are also different in that they allow for a wide range of variation in the values of the Greeks and thus can more closely mirror the behavior observed in real-life portfolios.

3.7 Applications of RSO.

1. **Capital-constrained firm writes an option to sell itself**

   Consider a situation in which one firm enters a contract to buy another firm, whose stock is traded, at the end of three years at 125% of an escalator value, which is an average of a benchmark price and the firm value at the time of exercising the option to purchase. Such a contract can be modeled in a straightforward manner as an RSO call option. The seller is concerned that if the company does well over the intervening three year-period that the purchase price should rise to reflect the increased value of the company. At the same time if the company falls on hard times it does not want to be acquired cheaply when it may simply be going through a temporary bad phase. So it sets an acquisition value tied to an average of $K = \$25$ million and the actual firm value. Assume that $\alpha = .5$ through negotiation.[8]

What is the value of such an option if $S(0) = \$50$ million is the (current market value of the target firm), $\sigma = .20$,(annualized asset price volatility) , $r = .08$ (annualized interest rate)? The RSO call option parameters corresponding to the details of this particular contract are $\lambda = 1.25, \beta = 1, \alpha = .5,\ \delta = 0, \tau = 3$ and K=25 million.

The value of risk-scaling call option is \$11.613 million. The benefits of writing such an option are immediate. A perhaps struggling firm that needs cash infusion to right itself, can acquire working capital funds, without giving up immediate ownership rights or taking on debt. Additionally if at the end of the 3 years the market value of the target is \$100 million, the acquisition price jumps to $1.25(25)^{.5}(100)^{.5} =$ \$62. 5 million. The acquirer buys the company worth \$100 million for \$62. 5 million but has given up \$11.613 million (3 years ago for the right to purchase the firm). Total effective acquisition costs are

[8] In these types of deals there are inherent moral hazards associated with making upfront payments to an incumbent management team.. To mitigate the moral hazard inherent in fixing a purchase price schedule *ex ante*, as part of the contract the prospective acquiring firm typically receives significant oversight rights with regard to managing the target firm during the intervening period. In our case there will be a set of λ, β and α that allays the purchaser's fears. We assume that the selected parameters resolve these issues.

$(11.613)\exp(.08(3)) + 62.5 = \$77.263\,016\,38$ million.

This represents a takeover premium relative to the initial value of the company but a discount relative to the present value of the company. The acquiring firm would not be unhappy. Neither is the target likely to be unhappy. They acquired working capital and after their company's fortunes improved the company was sold for \$62.5 million. Both parties unequivocally benefit from this agreement.

2. **Joint Venture**

A firm wants to develop a project, for which the development costs are currently estimated at \$200 million. The time to development is estimated at one year during which time the costs may vary randomly. The estimated present value of the gross revenues of the project ($S(0)$) is \$300 million (since the project is not a traded asset its value must be an estimate.) The revenues themselves are subject to revision as the project comes on stream. However, the firm that owns the development rights cannot fund the project on its own. It decides to write an option to a partner firm that would fund 40% of the development costs, in exchange for 75% of the project's revenues. Let $K = \$200$ (million) and the actual development costs evolve according to $(K)^{.2}(S_T)^{.8}$.

Assume that $\sigma = .20$,(annualized asset price volatility) , $r = .08$ (annualized interest rate). To price this option we need to also set $\lambda = .40,\ \delta = 0, \beta = .75, \alpha = .8, \tau = 1$. Let $S_T = \$500$ (million) when the project is brought on stream. The value of this option is \$116.4513216 (million) which is to be paid to the original developer. The revised costs of the project are

$$(200)^{.2}(500)^{.8} = \$416.\,276\,603\,7$$

The new partner's share of the project development costs will be

$$.4(200)^{.2}(500)^{.8} = \$166.\,510\,641\,5$$

The net proceeds to new partner are

$$(.75)500 - (116.4513216)\exp(.08) - 166.\,510\,6415$$

$= 82.\,339\,147\,80$ million.

The net proceeds to original developer are

$$(.25)500 + (116.4513216)\exp(.08) - .6(200)^{.2}(500)^{.8}$$

$=\$\ 1.\,384\,248\,482$ millions.

Both partners benefit if the project comes on stream. Note that if both parties agree that the option premium will only be paid if the project is implemented then the resulting relative payoffs are unchanged. [9]

[9] We see that the mutual benefits from the project are unchanged with respect to the

Changing the parameters slightly changes the resulting allocation of net benefits. Let $\lambda = .40,\ \delta = 0, \beta = .60, \alpha = .8, \tau = 1$. Let $S_T = \$500$ (millions) when the project is brought on stream. The value of this option is \$71.4513216 million.

Net proceeds to new partner

$=(.60)500 - (71.4513216)\exp(.08) - 166.510\,641\,5 = 56.087\,065\,84$

Net proceeds to original developer=

$(.40)500 + (71.4513216)\exp(.08) - .6(200)^{.2}(500)^{.8} = 27.636\,330\,44$

Our model can therefore price the value of any such development option provided that the development costs do not exceed the gross flows from the project. Such cases are of no interest as no rational set of developers would be interested in such projects. If the revenues of the projects fall and development costs fall also we can price such options.

3.8 Greeks For RSOs

Since RSOs span a much greater space than standard options they obey rules that are more generalized than those of fixed exercise price options. We present the equivalent forms for RSOs and then show that they specialize

timing of the payment of the option premium. Pay-later options have similar properties. These options have the feature that the option premium is paid only if the option is in the money at the time of exercise. At exercise both counterparties receive payments. Deep in the money standard call options have the same effective prices as pay-later call options with equivalent parameters.

to known results. The results below are valid for RSOs with parameters satisfying the restrictions imposed in section 3. Derivations of the results presented in this section are described in Appendix A.

In addition to the standard Greeks we present three new measures of option sensitivity. We term the sensitivity of the option price to the α parameter the option's *warp*. In the model the variable λ is a coverage measure. If the option is a call (put) option λ determines the cost (revenue) coverage a party assumes in acquiring (disposing of) the asset or a portion thereof. The higher (lower) the degree of coverage the lower (higher) the value of the call (put). We term the response of the option price to changes in λ the option's λ-*gearing.*

Finally we examine the response of the value of the option to changes in β . In the model the variable β also determines the benefits (and costs) accruing to a party from exercise of the option, in the case of both call and put options. If the option is a call (put) option β determines the revenue (costs) a party assumes in acquiring (disposing of) the asset or a portion thereof. The higher (lower) the degree of coverage the higher (lower) the value of the call (put). We term the response of the option price to changes in β the option's β-*gearing.*

3.9 The "Greeks"

Here we present the comparative static results for the call and the put version of the RSO model. We also show precisely the relation between the standard

Black-Scholes type results and those of the new model. This allows us to assess the impact of scaling on the sensitivities of the call and the put options. As discussed earlier α, β and λ effect scaling in different ways.

Warp

$$\begin{aligned}\frac{\partial C}{\partial \alpha} &= \delta\lambda\tau e^{-\alpha\delta\tau}\Upsilon N(d_2) - \lambda S e^{-\alpha\delta\tau}\sigma\sqrt{\tau}\Upsilon n(d_2) \\ &- \Upsilon N(d_2)(\ln(S/K) + (r\tau + \alpha\sigma^2\tau - \sigma^2\tau/2))\lambda e^{-\alpha\delta\tau} < 0\end{aligned}$$

$$\begin{aligned}\frac{\partial P}{\partial \alpha} &= -\delta\lambda\tau e^{-\alpha\delta\tau}\Upsilon(1 - N(d_2)) - \sigma\sqrt{\tau}\beta S e^{-\delta\tau} n(d_1) \\ &+\lambda e^{-\alpha\delta\tau}\{\Upsilon\ln(S/K) + (r + (\alpha\sigma^2 - .5\sigma^2))\tau\}(1 - N(d_2)) < 0\end{aligned}$$

Delta

$$\frac{\partial C}{\partial S} = \frac{C}{S} + (1-\alpha)\lambda e^{-\alpha\delta\tau}(\frac{\Upsilon}{S})N(d_2) > 0$$

$$\frac{\partial P}{\partial S} = \frac{P}{S} - (1-\alpha)\lambda e^{-\alpha\delta\tau}(\frac{\Upsilon}{S})(1 - N(d_2)) < 0$$

Gamma

$$\frac{\partial^2 C}{\partial S^2} = \beta n(d_1)e^{-\delta\tau}(S\sigma\sqrt[2]{\tau})^{-1} - \lambda(\alpha)(\alpha-1)e^{-\alpha\delta\tau}\Upsilon S^{-2}N(d_2) - \alpha\beta n(d_1)e^{-\delta\tau}(\sigma\sqrt{\tau}S)^{-1} > 0$$

$$\frac{\partial^2 P}{\partial S^2} = \beta n(d_1)e^{-\delta\tau}(S\sigma\sqrt[2]{\tau})^{-1} + \lambda(\alpha)(\alpha-1)e^{-\alpha\delta\tau}\Upsilon S^{-2}(1-N(d_2)) - \alpha\beta n(d_1)e^{-\delta\tau}(\sigma\sqrt{\tau}S)^{-1}$$

may be positive or negative.

Vega

$$\frac{\partial C}{\partial \sigma} = (1-\alpha)\sqrt[2]{\tau}\beta S e^{-\delta\tau}n(d_1) - \lambda(\alpha)(\alpha-1)\sigma\tau e^{\alpha\delta\tau}\Upsilon N(d_2) > 0$$

$$\frac{\partial P}{\partial \sigma} = (1-\alpha)\sqrt[2]{\tau}\beta S e^{-\delta\tau}n(d_1) + \lambda(\alpha)(\alpha-1)\sigma\tau e^{\alpha\delta\tau}\Upsilon(1-N(d_2))$$

may be positive or negative.

Theta

$$\begin{aligned}\frac{\partial C}{\partial \tau} &= -\delta\beta S N(d_1)e^{-\delta\tau} + \alpha\lambda\delta\Upsilon e^{-\alpha\delta\tau}N(d_2) \\ &\quad + (1-\alpha)\{r + .5\alpha\sigma^2\}\lambda\Upsilon e^{-\alpha\delta\tau}N(d_2) + \beta S(1-\alpha)\sigma n(d_1)e^{-\delta\tau}(2\sqrt[2]{\tau})^{-1}\end{aligned}$$

$$\begin{aligned}\frac{\partial P}{\partial \tau} &= \beta S(1-\alpha)\sigma n(d_1)e^{-\delta\tau}(2\sqrt[2]{\tau})^{-1} - \alpha\lambda\delta\Upsilon e^{-\alpha\delta\tau}(1-N(d_2)) \\ &\quad -\lambda(1-\alpha)\{r+.5\alpha\sigma^2\}\Upsilon e^{-\alpha\delta\tau}(1-N(d_2)) + \delta\beta S e^{-\delta\tau}(1-N(d_1))\end{aligned}$$

<u>Rho</u>

$$\frac{\partial C}{\partial r} = \lambda(1-\alpha)\tau e^{-\alpha\delta\tau}\Upsilon N(d_2) > 0$$

$$\frac{\partial P}{\partial r} = -\lambda(1-\alpha)\tau e^{-\alpha\delta\tau}\Upsilon(1-N(d_2)) < 0$$

<u>λ-Gearing</u>

$$\frac{\partial C}{\partial \lambda} = -e^{-\alpha\delta\tau}\Upsilon N(d_2) < 0$$

$$\frac{\partial P}{\partial \lambda} = e^{-\alpha\delta\tau}\Upsilon(1-N(d_2)) > 0$$

<u>β-Gearing</u>

$$\frac{\partial C}{\partial \beta} = S e^{-\delta\tau}N(d_1) > 0$$

$$\frac{\partial P}{\partial \beta} = -S e^{-\delta\tau}(1-N(d_1)) < 0$$

<u>Kappa</u>

$$\frac{\partial C}{\partial K} = -\lambda(1-\alpha)e^{-\alpha\delta\tau}(\frac{\Upsilon}{K})N(d_2) < 0$$

$$\frac{\partial P}{\partial K} = \lambda(1-\alpha)e^{-\alpha\delta\tau}(\frac{\Upsilon}{K})(1-N(d_2)) > 0$$

3.9.1 Absence of Symmetry

Unlike the standard Black-Scholes model, RSO approach generates comparative static results for option sensitivities that do not maintain any symmetry. Symmetry holds in the Black-Scholes world, for gamma and vega, since the put and call values of these measures are pair-wise equal. We show that as $\alpha \mapsto 0$, both the call and the put values of gamma and of vega are equal but that this is not true for the general $\alpha \neq 0$ case.

$>$From Section 3.6 above,we note that

$$\Gamma^{RSO}_{Call} = \Gamma^{BS}_{Call} - (\alpha)(\alpha-1)e^{-\alpha\delta\tau}\Upsilon S^{-2}N(d_2) - \alpha\beta n(d_1)e^{-\delta\tau}(\sigma\sqrt[2]{\tau})^{-1}$$

$$\Gamma^{RSO}_{Put} = \Gamma^{BS}_{Put} + (\alpha)(\alpha-1)e^{-\alpha\delta\tau}\Upsilon S^{-2}(1-N(d_2)) - \alpha\beta n(d_1)e^{-\delta\tau}(\sigma\sqrt[2]{\tau})^{-1}$$

where the Black Scholes gamma is $\beta n(d_1)e^{-\delta\tau}(S\sigma\sqrt[2]{\tau})^{-1}$. It is clear that as $\alpha \mapsto 0$,

$$\Gamma^{RSO}_{Call} = \Gamma^{BS}_{Call} = \Gamma^{RSO}_{Put} = \Gamma^{BS}_{Put}.$$

A similar argument applies for the case of option vegas.

$$Vega^{RSO}_{Call} = Vega^{BS}_{Call} - \alpha\sqrt[2]{\tau}\beta S e^{-\delta\tau} n(d_1) - \lambda(\alpha)(\alpha-1)\sigma\tau e^{\alpha\delta\tau}\Upsilon N(d_2)$$

$$Vega^{RSO}_{Put} = Vega^{BS}_{Put} - \alpha\sqrt[2]{\tau}\beta S e^{-\delta\tau} n(d_1) + \lambda(\alpha)(\alpha-1)\sigma\tau e^{\alpha\delta\tau}\Upsilon(1 - N(d_2))$$

where the Black-Scholes vega is $\sqrt{\tau}\beta S \exp^{-\delta\tau} n(d_1)$. We note that

$Vega^{RSO}_{Call} = Vega^{BS}_{Call} = Vega^{RSO}_{Put} = Vega^{BS}_{Put}$ whenever $\alpha = 0$.

Some additional features of RSOs need to be mentioned. Gammas and vegas in a Black-Scholes world can never turn negative even though they can approach zero arbitrarily close. In the case of puts, RSO gammas and vegas span both negative and positive values. It is very clear that in the case of deep in the money put options that option vegas and gammas can become negative. Consider the case of standard call and put options, where $\sigma = .20$, $r = .08$, $\lambda = 1$, $\delta = 0, \beta = 1, \alpha = 0, \tau = 1$, $S_T = \$30$ and $K = \$200$.

Option Type	Price	Gamma	Vega
Call	0	1.95017E-19	3.5103E-17
Put	154.62326	1.95017E-19	3.5103E-17

However for an RSO with parameters $\sigma = .20$, $r = .08$,$\lambda = 1$, $\delta = 0, \beta = 1$, $\alpha = .5, \tau = 1$, $S_T = \$30$ and $K = 200$ the picture changes dramatically.

Option Type	Price	Gamma	Vega
Call	0	9.75084E-20	1.75515E-17
Put	44.05125	-.0205697	-3.70256

In essence when the RSO option has a strike price substantially below the asset price increases in the underlying volatility actually reduces the value of the put version of the option. This occurs because the effective risk adjusted strike price falls and makes holding the put option in isolation a less attractive proposition. However for portfolio managers this is potentially very useful. If a portfolio has positive vega then instead of going short standard put options to reduce volatility risk or to make the portfolio vega neutral a portfolio manager can go long RSOs puts and achieve the same result. Going long a put is inherently less risky than shorting a put and the RSO put confers a natural advantage in this case.

4 Conclusion

We derive closed form solutions for a model of option pricing with exercise price uncertainty arising through dependence upon the price of the underlying asset at expiration. We show that a European option with this particular form of exercise price uncertainty is equivalent to a portfolio of options with fixed exercise prices. All comparative statics are provided and we show the equivalence of our results in the case of $\alpha = 0$ with those of the Black-Scholes model. We show that option Gammas and Vegas just like those of portfolios can be negative.

In addition, we illustrate how RSOs can be used to realistically model contracts in a variety of corporate finance applications in which exercise price uncertainty is resolved at contract maturity. The choice parameters allow contract counterparties to scale potential payoffs and thus fix their expected risk exposure as desired. We apply our pricing approach to value certain types of real-options involved in joint ventures and the raising of capital. We believe that the RSO approach permits, in principle, the easy valuation of non-traded options with these features since the only additional information needed is known at the outset.

4.1 Appendix A

In order to verify the presented results on the derivation of the Greeks, some relations are helpful. We derive all the results for the call options. The results for the put version of the model are easily derived by appealing to the put call parity relation for the RSO class of options.

(i) $d_2 = d_1 - (1-\alpha)\sigma\sqrt[2]{\tau}$

(ii) $\frac{\partial d_1}{\partial \tau} = \frac{(r-\delta)\sigma\sqrt{\tau}+.5\sigma^2(\sigma\sqrt{\tau}-d_1)}{\sigma^2\tau}$

(iii) $\frac{\partial d_2}{\partial \tau} = \frac{\partial d_1}{\partial \tau} - \frac{.5(1-\alpha)\sigma}{\sqrt[2]{\tau}}$

(iv) $n(d_2)\Upsilon = n(d_1)\frac{\beta S \exp(-(1-\alpha)\delta\tau)}{\lambda}$

(v) $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt[2]{\tau}}$

Warp

$$\begin{aligned}\frac{\partial C}{\partial \alpha} &= \delta\lambda\tau e^{-\alpha\delta\tau}\Upsilon N(d_2) - \lambda S e^{-\alpha\delta\tau}\sigma\sqrt{\tau}\Upsilon n(d_2)\\ &- \Upsilon N(d_2)(\ln(S/K) + (r\tau + \alpha\sigma^2\tau - \sigma^2\tau/2))\lambda e^{-\alpha\delta\tau} < 0\end{aligned}$$

Delta

$\frac{\partial C}{\partial S} = \beta N(d_1)e^{-\delta\tau} + \beta S n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial S} - \lambda\Upsilon e^{-\alpha\delta\tau}n(d_2)\frac{\partial d_1}{\partial S} - \alpha\lambda\frac{\Upsilon}{S}e^{-\alpha\delta\tau}N(d_2)$

But $\beta S n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial S} - \lambda\Upsilon e^{-\alpha\delta\tau}n(d_2)\frac{\partial d_1}{\partial S} = 0$ using (iv) and (i).

Therefore

$\frac{\partial C}{\partial S} = \beta N(d_1)e^{-\delta\tau} - \alpha\lambda\frac{\Upsilon}{S}e^{-\alpha\delta\tau}N(d_2)$.

The next result follows by substituting for the value of the RSO Call.

$\frac{\partial C}{\partial S} = \frac{C}{S} + (1-\alpha)\lambda e^{-\alpha\delta\tau}(\frac{\Upsilon}{S})N(d_2) > 0$

Gamma

$\frac{\partial^2 C}{\partial S^2} = \beta n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial S} - \lambda(\alpha)(\alpha-1)e^{-\alpha\delta\tau}S^{-2}\Upsilon N(d_2) - \alpha\lambda\frac{\Upsilon}{S}e^{-\alpha\delta\tau}n(d_2)\frac{\partial d_2}{\partial S}$.

Apply (iv) and (v) and the result below follows.

$\frac{\partial^2 C}{\partial S^2} = \beta n(d_1)e^{-\delta\tau}(S\sigma\sqrt[2]{\tau})^{-1} - \lambda(\alpha)(\alpha-1)e^{-\alpha\delta\tau}S^{-2}\Upsilon N(d_2) - \alpha\beta n(d_1)e^{-\delta\tau}(S\sigma\sqrt[2]{\tau})^{-1}$

Vega

$\frac{\partial C}{\partial \sigma} = \beta S n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial \sigma} - \lambda\Upsilon e^{-\alpha\delta\tau}n(d_2)\frac{\partial d_2}{\partial \sigma} - \alpha(\alpha-1)\sigma\tau\lambda\Upsilon e^{-\alpha\delta\tau}N(d_2)$

$=\beta S n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial \sigma} - \lambda\Upsilon e^{-\alpha\delta\tau}n(d_2)(\frac{\partial d_1}{\partial \sigma} - (1-\alpha)\sqrt[2]{\tau}) - \alpha(\alpha-1)\sigma\tau\lambda\Upsilon e^{-\alpha\delta\tau}N(d_2)$

$=\beta S n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial \sigma} - \lambda e^{-\alpha\delta\tau}n(d_1)\frac{\beta S\exp(-(1-\alpha)\delta\tau)}{\lambda}(\frac{\partial d_1}{\partial \sigma} - (1-\alpha)\sqrt[2]{\tau}) - \alpha(\alpha-$
$1)\sigma\tau\lambda\Upsilon e^{-\alpha\delta\tau}N(d_2)$

by (iv).

Now $\beta S n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial \sigma} - \lambda e^{-\alpha\delta\tau}n(d_1)\frac{\beta S\exp(-(1-\alpha)\delta\tau)}{\lambda}(\frac{\partial d_1}{\partial \sigma}) = 0$

Hence by further simplification

$\frac{\partial C}{\partial \sigma} = (1-\alpha)\sqrt[2]{\tau}\beta S e^{-\delta\tau}n(d_1) - \lambda(\alpha)(\alpha-1)\sigma\tau e^{-\alpha\delta\tau}\Upsilon N(d_2)$ (positive or negative)

Theta

$\frac{\partial C}{\partial \tau} = -\delta\beta S N(d_1)e^{-\delta\tau} + \alpha\lambda\delta\Upsilon e^{-\alpha\delta\tau}N(d_2) + (1-\alpha)\{r + .5\alpha\sigma^2\}\lambda\Upsilon e^{-\alpha\delta\tau}N(d_2)$

$+\beta S n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial \tau} - \lambda\Upsilon e^{-\alpha\delta\tau}n(d_2)(\frac{\partial d_2}{\partial \tau})$.

But note that

$\beta S n(d_1)e^{-\delta\tau}\frac{\partial d_1}{\partial \tau} - \lambda\Upsilon e^{-\alpha\delta\tau}n(d_2)(\frac{\partial d_2}{\partial \tau}) = \beta S(1-\alpha)\sigma n(d_1)e^{-\delta\tau}(2\sqrt[2]{\tau})^{-1}$

by virtue of (iv) and (iii). The result below then follows.

$\frac{\partial C}{\partial \tau} = -\delta\beta S N(d_1)e^{-\delta\tau} + \alpha\lambda\delta\Upsilon e^{-\alpha\delta\tau}N(d_2) + (1-\alpha)\{r + .5\alpha\sigma^2\}\lambda\Upsilon e^{-\alpha\delta\tau}N(d_2)$

$+\beta S(1-\alpha)\sigma n(d_1)e^{-\delta\tau}(2\sqrt[2]{\tau})^{-1} > 0(always)$

Rho

$\frac{\partial C}{\partial r} = -\lambda(\alpha-1)\tau e^{-\alpha\delta\tau}\Upsilon N(d_2)$ (positive)

$\lambda - gearing$

$\frac{\partial C}{\partial \lambda} = -e^{-\alpha\delta\tau}\Upsilon N(d_2) < 0$

$\beta - gearing$

$\frac{\partial C}{\partial \beta} = S e^{-\delta\tau}N(d_1) > 0$

Kappa

$\frac{\partial C}{\partial K} = -\lambda(1-\alpha)e^{-\alpha\delta\tau}(\frac{\Upsilon}{K})N(d_2) < 0$

References

[1] Biger, N., and Hull, J. (1983) The Valuation of Currency Options, Financial Management:24-27

[2] Black, F., and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities, Journal of Political Economy81:637-654

[3] Boyle, P. P., and Turnbull, S. M.(1989) Pricing and Hedging Capped Options,Journal of Futures Markets, 9:41-54

[4] Briys, E., and Crouhy, M. (1988) Creating and Pricing Hybrid Foreign Currency Options, Financial Management:59-64

[5] Carr, P. (1988) The Valuation of Sequential Exchange Opportunities, Journal of Finance, 43:1235-1256

[6] Esser, A. (2003) General Valuation Principles For Arbitrary Payoffs and Applications to Power Options under Stochastic Volatility,Financial Markets and Portfolio Management,17:351-372.

[7] Fischer, S. (1978) Call Option Pricing when the Exercise Price is Uncertain, and the Valuation of Index BondsJournal of Finance,33:169-176

[8] Galai, D. (1983) Pricing of Optional Bonds, Journal of Banking and Finance,7:323-337

[9] Geske, R. (1979) The Valuation of Compound Options,Journal of Financial Economics7:63-81

[10] Grabbe, J. O. (1983) The Pricing of Call and Put Options on Foreign Exchange, Journal of International Money and Finance2:239-253

[11] Grabbe, J.O. (1987) Foreign Currency Options: A Survey in "Recent Developments in International Banking and Finance" Vol: 1: Sarkis J. Khoury and Alo Gosh eds.Lexington Books.

[12] Gu, A.Y. (2002) Valuing the Option to Purchase an Asset at a Proportional Discount, Journal of Financial Research, Vol. 25, 99-109.

[13] Johnson, H. (1987) Options on the Maximum or the Minimum of Several AssetsJournal of Financial and Quantitative Analysis 22:277-283

[14] Kemna A.G.Z., and Vorst, A.C.F. (1990) A Pricing Method for Options Based on Average Asset ValuesJournal of Banking and Finance14:113-129

[15] Margrabe, W. (1978) The Value of an Option to Exchange One Asset for Another, Journal of Finance33:177-186

[16] Merton, R. C. (1973) Theory of Rational Option Pricing, Bell Journal of Economics and Management Science4:141-183

[17] Ritchken, P., Sankarasubrahmaniam, L. and Vijh, A. M. (1990, Autumn) Averaging Options for Capping Total Costs, Financial Management:35-41

[18] Pindyck, R. S (1993) Investments of Uncertain Costs, Journal of Financial Economics, Vol. 34, 53-76.

[19] Rumsey, J. (1991) Pricing Cross-Currency Options, Journal of Futures Markets, 11:89-93

[20] Stulz, R. M. (1982) Options on the Minimum or The Maximum of Two Risky Assets, Journal of Financial Economics10:161-185

[21] Zhang, P. G. (1997) *Exotic Options: A Guide to the Second Generation Options*, Worldwide Scientific