Power Exchange Options

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Abstract

In this paper we present pricing results for an option to exchange the value of one asset raised to a power ($S_1^{\alpha_1}$) for the value of another asset raised to a power ($S_2^{\alpha_2}$). We refer to such options as *power exchange options* since they simultaneously generalize results for both the Fischer-Margrabe-type option to exchange one asset for another and power options. We explicitly solve for the price of the European power exchange option under the assumption of risk-neutrality. We also use our results to price options paying the best or worst of powers of two assets. Finally, we establish sufficient conditions for the equivalence of the pricing problems of the American and European power exchange options.

Key words: Exotic Options, Exchange Options, Power Options

1 Introduction

Independently Fischer (1978) and Margrabe (1978) derive pricing formulas under the assumption of risk-neutrality for options with payoffs of the form

$$(S_1(T) - S_2(T))^+ .^{1} \tag{1}$$

In the Fischer approach, S_2 is interpreted as a stochastic exercise price at which an asset S_1 may be purchased, while Margrabe interpreted the contract as an option to exchange one asset for another. The slight difference in appearance between their pricing formulas is due to the fact that Fischer allows for the possibility that the stochastic exercise price is not the price of a traded asset. Thus it is necessary

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[1] For a given function f, the notation f^+ refers to the positive part function and is defined by $f(x)^+ := \max\{f(x), 0\}$

to determine the market price of exercise price risk in order to apply Fischer's formula. In Margrabe's version, it is assumed that both processes are the prices of traded assets.

In this paper we investigate a generalization of the Fischer-Margrabe-type option. Specifically, we present results on pricing options whose payoffs are of the form

$$\phi(S_1(T), S_2(T)) := (\lambda_1 S_1^{\alpha_1}(T) - \lambda_2 S_2^{\alpha_2}(T))^+. \tag{2}$$

We refer to contracts possessing these payoffs as *power exchange options*. Such options have not been previously studied in the literature, despite the fact that they represent a simultaneous generalization of Fischer-Margrabe exchange options as well as power options, both of which have found a number of useful applications. [2] Exchange options have proven particularly useful in the field of compensation design where executive stock option compensation depends on the relative performance of a company's stock compared to a benchmark index (see, for example, Johnson and Tian (2000)). Tompkins (1999) discusses several applications of power options in hedging nonlinear risks, such as those arising in option positions from changing levels of implied volatility.

In the area of compensation design, power exchange options could be used to capture the efficiencies of indexing while allowing for finer control of the incentive aligning properties through more flexibility in parameterizing the contract. We show this by demonstrating that power exchange options provide a natural generalization of the indexed executive stock options studied in Johnson and Tian (2000).

We explicitly solve for the price of the European power exchange option under the assumption of risk-neutrality. Analogously to the case of standard exchange op-

[2] See Zhang (1997) or Tompkins (1999) for pricing results for power options.

tions, we also apply our results to price options paying the best or worst of powers of two assets. Finally, we generalize the results of Margrabe (1978) for American exchange options by establishing sufficient conditions for the equivalence of the pricing problems of the American and European power exchange options.

2 Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. We define a *market environment* consisting of two risky assets S_1 and S_2, and a risk-free bond B. The prices of these assets are assumed to be governed by the stochastic differential equations,

$$dS_i(t) = (\mu_i - \delta_i)S_i(t)\,dt + \sigma S_i(t)dB_i(t), \quad t \in [0,T], \tag{3}$$

where for $i = 1, 2$, $\mu_i, \delta_i \in \mathbb{R}$ (the set of real numbers), $\sigma_i \in \mathbb{R}^+$ (the set of positive real numbers), B_i is a standard Brownian motion under P, $dB_1\,dB_2 = \rho\,dt$ and

$$dS_0(t) = r\,dt, \quad t \in [0,T], \tag{4}$$

where $r \in \mathbb{R}^+$.

It is convenient to model the correlation between B_1 and B_2 by defining

$$\begin{aligned} &dS_1(t) = (\mu_1 - \delta_1)S_1(t)\,dt + \sigma_1 S_1(t)dB_1(t) \\ &dS_2(t) = (\mu_2 - \delta_2)S_2(t)\,dt + \sigma_2\rho S_2(t)dB_1(t) + \sigma_2\sqrt{1-\rho^2}S_2(t)dB_2(t), \quad t \in [0,T], \end{aligned} \tag{5}$$

where B_1 and B_2 are independent Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

As is well-known, there are no arbitrage opportunities in such a market environment if and only if there exists a probability measure $\widetilde{P}$ under which S_i/S_0 is a martingale for $i = 1, 2$. On the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \widetilde{P})$ equations (5) and (4) become

$$dS_1(t) = (r-\delta_1)S_1(t)\,dt + \sigma_1 S_1(t)d\widetilde{B}_1(t)$$
$$dS_2(t) = (r-\delta_2)S_2(t)\,dt + \sigma_2\rho S_2(t)d\widetilde{B}_1(t) + \sigma_2\sqrt{1-\rho^2}S_2(t)d\widetilde{B}_2(t), \quad t\in[0,T], \tag{6}$$

where, for all $t \in [0,T]$

$$\widetilde{B}_1(t) = B_1(t) + ((\mu_1-\delta_1-r)/\sigma_1)t$$

and

$$\widetilde{B}_2(t) = B_2(t) + ((\mu_2-\delta_2-r-\rho(\sigma_1/\sigma_2)(\mu_1-\delta_1-r))/(\sigma_2\sqrt{1-\rho^2}))t$$

are standard Brownian motions under $\widetilde{P}$, and

$$dS_0(t) = r\,dt, \quad t\in[0,T]. \tag{7}$$

A frictionless environment is posited where there are no constraints on short sales and there are no transactions costs attendant upon purchase or sale of the underlying assets or the option.

By a *portfolio* in a financial market environment, we mean a 3-dimensional stochastic process $(\pi_0(\cdot),\pi_1(\cdot),\pi_2(\cdot))$ in which $\pi_0(\cdot)$ and $\pi_i(\cdot)$, $i=1,2$ are $\{\mathcal{F}(t)\}$-adapted, progressively measurable processes with $\pi_i(\cdot)$, $i=0,1,2$ satisfying the usual integrability requirements (cf. Karatzas and Shreve (1998) page 7). Here, π_0 represents the market value of the bonds held in the portfolio, while π_i represents the market value of the position in risky asset S_i.

Theorem 1 *The value at time t of a European power exchange option with parameters $\xi=(r,\alpha_1,\alpha_2,\lambda_1,\lambda_2,\delta_1,\delta_2)$, and expiration time $T>t$ is*

$$PE(t,S_1,S_2,\xi;T) = \Upsilon_1(t,S_1,\xi;T)\,N(d_1) - \Upsilon_2(t,S_2,\xi;T)\,N(d_2) \tag{8}$$

where

$$d_1 = \frac{\ln\left(\frac{\lambda_1 S_1^{\alpha_1}}{\lambda_2 S_2^{\alpha_2}}\right) + \left[\alpha_1(r-\delta_1) - \alpha_2(r-\delta_2) - \alpha_1(1-\alpha_1)\frac{\sigma_1^2}{2} + \alpha_2(1-\alpha_2)\frac{\sigma_2^2}{2} + \frac{1}{2}v^2\right](T-t)}{v\sqrt{(T-t)}}$$

$$d_2 = \frac{\ln\left(\frac{\lambda_1 S_1^{\alpha_1}}{\lambda_2 S_2^{\alpha_2}}\right) + \left[\alpha_1(r-\delta_1) - \alpha_2(r-\delta_2) - \alpha_1(1-\alpha_1)\frac{\sigma_1^2}{2} + \alpha_2(1-\alpha_2)\frac{\sigma_2^2}{2} - \frac{1}{2}v^2\right](T-t)}{v\sqrt{(T-t)}}$$

and

$$v^2 = \alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 - 2\alpha_1\alpha_2\sigma_1\sigma_2\rho$$

$$\Upsilon_1(t, S_1, \xi; T) = \lambda_1 S_1^{\alpha_1} \exp\{[(\alpha_1 - 1)r - \alpha_1\delta_1 - \alpha_1(1-\alpha_1)\frac{\sigma_1^2}{2}](T-t)\}$$

$$\Upsilon_2(t, S_2, \xi; T) = \lambda_2 S_2^{\alpha_2} \exp\{[(\alpha_2 - 1)r - \alpha_2\delta_2 - \alpha_2(1-\alpha_2)\frac{\sigma_2^2}{2}](T-t)\}.$$

$N(\cdot)$ *denotes the cumulative normal density function.*

PROOF. Since neither $S_1^{\alpha_1}$ nor $S_2^{\alpha_2}$ is the price of a traded asset, we cannot simply make use of a change of numerairé technique. However, it is possible to adapt the general change of measure technique discussed in Esser (2003) for use in the present situation. Specifically, we define a measure $P^{(\alpha_2)}$ equivalent to $\widetilde{P}$ by the Radon-Nikodým derivative

$$dP^{(\alpha_2)}/d\widetilde{P} = \frac{S_2^{\alpha_2}(T)}{\widetilde{E}_0[S_2^{\alpha_2}(T)]}.$$

Note that

$$S_2^{\alpha_2}(t) = S_2^{\alpha_2}(0)\exp\{(\alpha_2(r-\delta_2) - \alpha_2\sigma_2^2/2)t + \alpha_2\sigma_2\rho\widetilde{B}_1(t) + \alpha_2\sigma_2\sqrt{1-\rho^2}\widetilde{B}_2(t)\},$$

and

$$\widetilde{E}_t[S_2^{\alpha_2}(T)] = S_2^{\alpha_2}(t)\exp\{(\alpha_2(r-\delta_2) - \alpha_2\sigma_2^2/2 + \alpha_2^2\sigma_2^2/2)(T-t)\},$$

so that

$$dP^{(\alpha_2)}/d\widetilde{P} = \exp\{-\alpha_2^2\sigma_2^2/2T + \alpha_2\sigma_2\rho\widetilde{B}_1(T) + \alpha_2\sigma_2\sqrt{1-\rho^2}\widetilde{B}_2(T)\}.$$

Now, the arbitrage-free price of the power exchange option is given by

$$
\begin{aligned}
PE(t,S_1,S_2,\xi;T) &= e^{-r(T-t)}\widetilde{E}_t[\phi(S_1(T),S_2(T))] \\
&= e^{-r(T-t)}\widetilde{E}_t[(\lambda_1 S_1^{\alpha_1}(T)-\lambda_2 S_2^{\alpha_2}(T))^+] \\
&= e^{-r(T-t)}\widetilde{E}_t\left[\lambda_2 S_2^{\alpha_2}(T)\left(\frac{\lambda_1 S_1^{\alpha_1}(T)}{\lambda_2 S_2^{\alpha_2}(T)}-1\right)^+\right] \\
&= e^{-r(T-t)}\widetilde{E}_t[\lambda_2 S_2^{\alpha_2}(T)]\widetilde{E}_t\left[\frac{\lambda_2 S_2^{\alpha_2}(T)}{\widetilde{E}_t[\lambda_2 S_2^{\alpha_2}(T)]}\left(\frac{\lambda_1 S_1^{\alpha_1}(T)}{\lambda_2 S_2^{\alpha_2}(T)}-1\right)^+\right] \\
&= e^{-r(T-t)}\widetilde{E}_t[\lambda_2 S_2^{\alpha_2}(T)]E_t^{(\alpha_2)}\left[\left(\frac{\lambda_1 S_1^{\alpha_1}(T)}{\lambda_2 S_2^{\alpha_2}(T)}-1\right)^+\right] \\
&= e^{-r(T-t)}\lambda_2 S_2^{\alpha_2}(t)\exp\{(\alpha_2(r-\delta_2)-\alpha_2\frac{\sigma_2^2}{2}+\alpha_2^2\frac{\sigma_2^2}{2})(T-t)\} \\
&\quad \times E_t^{(\alpha_2)}\left[\left(\frac{\lambda_1 S_1^{\alpha_1}(T)}{\lambda_2 S_2^{\alpha_2}(T)}-1\right)^+\right] \qquad (9)
\end{aligned}
$$

The conditional expectation in the final line above can be calculated explicitly. Observe that under $P^{(\alpha_2)}$,

$$
\begin{aligned}
Y^{(\alpha_2)} &:= \frac{S_1^{\alpha_1}(t)}{S_2^{\alpha_2}(t)} \\
&= \frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)}\exp\{(\alpha_1(r-\delta_1)-\alpha_1\frac{\sigma_1^2}{2}-\alpha_2(r-\delta_2)+\alpha_2\frac{\sigma_2^2}{2}-\alpha_2^2\sigma_2^2+\alpha_1\alpha_2\sigma_1\sigma_2\rho)t \\
&\quad +\alpha_1\sigma_1\widetilde{B}_1^{(\alpha_2)}(t)-\alpha_2\sigma_2\rho\widetilde{B}_1^{(\alpha_2)}(t)-\alpha_2\sigma_2\sqrt{1-\rho^2}\widetilde{B}_2^{(\alpha_2)}(t)\},
\end{aligned}
\qquad (10)
$$

where $\widetilde{B}_1^{(\alpha_2)}(t)=-\alpha_2\sigma_2\rho t+\widetilde{B}_1(t)$ and $\widetilde{B}_2^{(\alpha_2)}(t)=-\alpha_2\sigma_2\sqrt{1-\rho^2}t+\widetilde{B}_2(t)$ are both Brownian motions by Girsanov's theorem. Furthermore, since $\widetilde{B}_1^{(\alpha_2)}(t)$ and $\widetilde{B}_2^{(\alpha_2)}(t)$ are independent, we can define the Brownian motion $W^{(\alpha_2)}$ by

$$
W^{(\alpha_2)}(t) := |v|^{-1}[(\alpha_1\sigma_1-\alpha_2\sigma_2\rho)\widetilde{B}_1^{(\alpha_2)}(t)-\alpha_2\sigma_2\sqrt{1-\rho^2}\widetilde{B}_2^{(\alpha_2)}(t)].
$$

Thus we can rewrite (10) as

$$
\begin{aligned}
Y^{(\alpha_2)} &= \frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)}\exp\{(\alpha_1(r-\delta_1)-\alpha_1\frac{\sigma_1^2}{2}-\alpha_2(r-\delta_2)+\alpha_2\frac{\sigma_2^2}{2}-\alpha_2^2\sigma_2^2+\alpha_1\alpha_2\sigma_1\sigma_2\rho)t \\
&\quad +\sqrt{\alpha_1^2\sigma_1^2+\alpha_2^2\sigma_2^2-2\alpha_1\alpha_2\sigma_1\sigma_2\rho}W^{(\alpha_2)}(t)\}.
\end{aligned}
$$

Now

$$E_t^{(\alpha_2)}\left[\left(\frac{\lambda_1 S_1^{\alpha_1}(T)}{\lambda_2 S_2^{\alpha_2}(T)}-1\right)^+\right]=\begin{cases}u(T-t,S_1^{\alpha_1}(t)/S_2^{\alpha_2}(t)), & t<T\\ \left(\frac{\lambda_1 S_1^{\alpha_1}(T)}{\lambda_2 S_2^{\alpha_2}(T)}-1\right)^+, & t=T,\end{cases}$$

where

$$u(\tau,y):=\int_{-\infty}^{\infty}\Big(\frac{\lambda_1}{\lambda_2}y\exp((\alpha_1(r-\delta_1)-\alpha_1\frac{\sigma_1^2}{2}-\alpha_2(r-\delta_2)+\alpha_2\frac{\sigma_2^2}{2}-\alpha_2^2\sigma_2^2+\alpha_1\alpha_2\sigma_1\sigma_2\rho)\tau + |v|z)-1)\Big)^+\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{z^2}{2\tau}}\,dz.$$

Since

$$\frac{\lambda_1}{\lambda_2}y\exp((\alpha_1(r-\delta_1)-\alpha_1\frac{\sigma_1^2}{2}-\alpha_2(r-\delta_2)+\alpha_2\frac{\sigma_2^2}{2}-\alpha_2^2\sigma_2^2+\alpha_1\alpha_2\sigma_1\sigma_2\rho)\tau+|v|z)>1$$

if and only if

$$z>z^*:=|v|^{-1}\Big[\ln(\frac{\lambda_1}{\lambda_2}y)-(\alpha_1(r-\delta_1)-\alpha_1\frac{\sigma_1^2}{2}-\alpha_2(r-\delta_2)+\alpha_2\frac{\sigma_2^2}{2} -\alpha_2^2\sigma_2^2+\alpha_1\alpha_2\sigma_1\sigma_2\rho)\tau\Big],$$

we can write

$$u(\tau,y)=\int_{z^*}^{\infty}\Big(\frac{\lambda_1}{\lambda_2}y\exp((\alpha_1(r-\delta_1)-\alpha_1\frac{\sigma_1^2}{2}-\alpha_2(r-\delta_2)+\alpha_2\frac{\sigma_2^2}{2}-\alpha_2^2\sigma_2^2 +\alpha_1\alpha_2\sigma_1\sigma_2\rho)\tau+|v|z)\Big)\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{z^2}{2\tau}}\,dz-\int_{z^*}^{\infty}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{z^2}{2\tau}}\,dz.$$

Changing variables by $z=-y\sqrt{\tau}+|v|\tau$ in the first integral and $z=-y\sqrt{\tau}$ in the second yields

$$E_t^{(\alpha_2)}\left[\left(\frac{\lambda_1 S_1^{\alpha_1}(T)}{\lambda_2 S_2^{\alpha_2}(T)}-1\right)^+\right]=u(T-t,S_1^{\alpha_1}(t)/S_2^{\alpha_2}(t))$$
$$=N(d_1)\exp\{(\alpha_1(r-\delta_1)-\alpha_2(r-\delta_2)-\alpha_1\frac{\sigma_1^2}{2}+\alpha_2\frac{\sigma_2^2}{2}+\alpha_1^2\frac{\sigma_1^2}{2}-\alpha_2^2\frac{\sigma_2^2}{2})(T-t)\}$$
$$-N(d_2).$$

Substituting this expression into (9), the result follows after simplifying.

Note that $\Upsilon_i(t,S_i,\xi;T)$, for $i=1,2$, in the pricing formula may be interpreted as the time t risk-neutral price of a forward contract to deliver $S_i^{\alpha_i}$ at time T. To see this, realize that

$$\begin{aligned}\Upsilon_i(t, S_i, \xi; T) &= e^{-r(T-t)} S_i^{\alpha_i}(t) \exp\{(\alpha_i(r - \delta_i) - \alpha_i \sigma_i^2/2 + \alpha_i^2 \sigma_i^2/2)(T - t)\} \\ &= e^{-r(T-t)} \widetilde{E}_t[S_i^{\alpha_i}(T)].\end{aligned} \tag{11}$$

It is straightforward to verify that familiar pricing formulas obtain as special cases of (8). For instance,

- For $\lambda_1 = 1, \alpha_1 = 1, \alpha_2 = 0, \lambda_2 \in \mathbb{R}^+$, (8) reduces to the familiar Black-Scholes formula for the price of a plain vanilla call with strike λ_2.
- For $\lambda_1 = 1, \alpha_1 \in \mathbb{R}^+, \alpha_2 = 0, \lambda_2 \in \mathbb{R}^+$, (8) becomes the price of a power call.
- For $\lambda_1 = \lambda_2 = 1, \alpha_1 = 1, \alpha_2 = 1$, (8) simplifies to the price of the standard exchange option.

Corollary 2 *The hedging portfolio for a short position in a European power exchange option with parameters* $\xi = (r, \alpha_1, \alpha_2, \lambda_1, \lambda_2, \delta_1, \delta_2)$*, and expiration time* $T > t$*, is given by*

$$\begin{aligned}&(\pi_0(t), \pi_1(t), \pi_2(t)) \\ &= (PE(t) - \alpha_1 N(d_1(t))\Upsilon_1(t) + \alpha_2 N(d_2(t))\Upsilon_2(t), \alpha_1 N(d_1(t))\Upsilon_1(t), \alpha_2 N(d_2(t))\Upsilon_2(t)).\end{aligned}$$

PROOF. For $i = 1, 2$,

$$\pi_i(t) = S_i(t) \frac{\partial}{\partial S_i} PE(t) = S_i \frac{(-1)^{i+1} \alpha_i N(d_i) \Upsilon_i}{S_i} = (-1)^{i+1} \alpha_i N(d_i) \Upsilon_i,$$

and $\pi_0(t) = PE(t) - \pi_1(t) - \pi_2(t)$.

Thus a perfect hedge is maintained by holding a long position in S_1 worth $\alpha_1 N(d_1)\Upsilon_1$, a short position in S_2 worth $\alpha_2 N(d_2)\Upsilon_2$, and either a long, short or zero position in the risk free asset depending on the sign of π_0. Equivalently, in view of (11), the same hedge is produced by a long position of $\alpha_1 N(d_1)$ forward contracts on $S_1^{\alpha_1}$, a short position of $\alpha_2 N(d_2)$ forward contracts on $S_2^{\alpha_2}$, and a position in the risk free asset as indicated above. Unlike the standard Margrabe case $(\alpha_1 = \alpha_2 = 1)$, hedging the power exchange option *does in general require* a non-zero position in the risk free asset. This is because the linear homogeneity in asset prices of the pricing formula in the standard case is not a characteristic of the general case.

We note in the following corollary that the results of Theorem 1 can be applied to derive the risk-neutral price of European options paying the best or worst of powers of two assets.

Corollary 3 *The value at time t of European options to deliver the best or worst of powers of two assets at expiration time $T > t$ are, respectively,*

$$PB(t, S_1, S_2; T) = \Upsilon_1(t, S_1, \xi; T)\, N(d_1) - \Upsilon_2(t, S_2, \xi; T)\, N(-d_2)$$

and

$$PW(t, S_1, S_2; T) = \Upsilon_1(t, S_1, \xi; T)\, N(-d_1) + \Upsilon_2(t, S_2, \xi; T)\, N(d_2)$$

PROOF. The result is then a straightforward consequence of (11) and the facts that

$$\max(\lambda_1 S_1^{\alpha_1}(T), \lambda_2 S_2^{\alpha_2}(T)) = (\lambda_1 S_1^{\alpha_1}(T) - \lambda_2 S_2^{\alpha_2}(T))^+ + \lambda_2 S_2^{\alpha_2}(T)$$

and

$$\min(\lambda_1 S_1^{\alpha_1}(T), \lambda_2 S_2^{\alpha_2}(T)) = \lambda_1 S_1^{\alpha_1}(T) - (\lambda_1 S_1^{\alpha_1}(T) - \lambda_2 S_2^{\alpha_2}(T))^+.$$

3 The American Power Exchange Option

Margrabe (1978) shows that, in the standard case, the value of the American exchange option with finite expiration time coincides with the value of the European version. That is, early exercise of the American exchange option is not optimal. The following result establishes sufficient conditions for the equivalence of the pricing problems of the American and European power exchange options.

Theorem 4 *If $\alpha_2 \leq 1$, $\alpha_1(r-\delta_1) \geq r$ and $\alpha_2(r-\delta_2) \leq r$, then for the American power exchange option with parameters $\xi = (r, \alpha_1, \alpha_2, \lambda_1, \lambda_2, \delta_1, \delta_2)$, and expiration time $T < \infty$, early exercise is not optimal, and its value is the same as that of the European version.*

PROOF. It suffices to show that the discounted payoff $e^{-rt}(\lambda_1 S_1^{\alpha_1}(t) - \lambda_2 S_2^{\alpha_2}(t))^+$ is a $\widetilde{P}$-submartingale (cf. Karatzas and Shreve (1998) Remark 5.11 pp. 59-60). For $0 \leq t \leq s \leq T$,

$$\begin{aligned}
\widetilde{E}_t[e^{-rs}(\lambda_1 S_1^{\alpha_1}(s) - \lambda_2 S_2^{\alpha_2}(s))^+] &\geq (\lambda_1 \widetilde{E}_t[e^{-rs} S_1^{\alpha_1}(s)] - \lambda_2 \widetilde{E}_t[e^{-rs} S_2^{\alpha_2}(s)])^+ \\
&= (\lambda_1 e^{(r(\alpha_1-\delta_1)-r)s} \widetilde{E}_t[e^{\alpha_1(\delta_1-r)s} S_1^{\alpha_1}(s)] - \lambda_2 e^{(r(\alpha_2-\delta_2)-r)s} \widetilde{E}_t[e^{\alpha_2(\delta_2-r)s} S_2^{\alpha_2}(s)])^+ \\
&\geq (\lambda_1 e^{(r(\alpha_1-\delta_1)-r)s}(e^{\alpha_1(\delta_1-r)t} S_1^{\alpha_1}(t)) - \lambda_2 e^{(r(\alpha_2-\delta_2)-r)s}(e^{\alpha_2(\delta_2-r)t} S_2^{\alpha_2}(t)))^+ \\
&= (\lambda_1 e^{(r(\alpha_1-\delta_1)-r)(s-t)}(e^{-rt} S_1^{\alpha_1}(t)) - \lambda_2 e^{(r(\alpha_2-\delta_2)-r)(s-t)}(e^{-rt} S_2^{\alpha_2}(t)))^+ \\
&\geq (e^{-rt}(\lambda_1 S_1^{\alpha_1}(t) - \lambda_2 S_2^{\alpha_2}(t)))^+,
\end{aligned}$$

where we have used Jensen's inequality in the first line. In passing from the second to the third line, we use the fact that for $i = 1, 2$, $e^{(\delta_i - r)t} S_i(t)$ is a $\widetilde{P}$-martingale and a convex (concave) function of a martingale is a submartingale (supermartingale). The final line follows from the fact that under the assumptions in the statement of the theorem, $e^{(r(\alpha_1-\delta_1)-r)t} \geq 1$ and $e^{(r(\alpha_2-\delta_2)-r)t} \leq 1$ for all $0 \leq t < \infty$.

4 Applications of Power Exchange Options

Among the possible applications of power exchange options, we mention the field of compensation design. The application to compensation design of the standard exchange option was first explored by Margrabe himself in his original paper in the context of incentive-based compensation for portfolio managers. There has recently been renewed interest in designing executive compensation contracts that reward performance measured relative to some benchmark (see, for example, Johnson and Tian (2000)). Such contracts have the potential for increasing the efficiency of compensation schemes by rewarding firm-specific performance only. Power exchange options can be used to capture the efficiencies of indexing while allowing for finer control of the incentive aligning properties through more flexibility in parameterizing the contract.

More explicitly, consider the exchange option proposed for use in executive compensation by Johnson and Tian (2000). They consider a contract having payoffs of the form $(P_T - H_T)^+$, where P is the company's stock price assumed to be a geometric Brownian motion $dP/P = (\mu_P - q_P)dt + \sigma_P dz_P$, and H is a benchmark stock that they define by

$$H_t = E[P_t | I_t,\, \alpha := \mu_P - r - \beta(\mu_I - r) = 0],$$

where I_t is some index assumed to be a geometric Brownian motion $dI/I = (\mu_I - q_I)dt + \sigma_I dz_I$, $dz_P dz_I = \rho dt$, $\beta = \rho(\sigma_P/\sigma_I)$, and α is interpreted to be the excess return of the stock relative to the index. Under these assumptions, they note that H is also a geometric Brownian motion and can be expressed in terms of β and I as

$$H_t = P_0 \left(\frac{I_t}{I_0}\right)^{\beta} e^{\eta t},$$

where

$$\eta = (r - q_P) - \beta(r - q_I) + \frac{1}{2}\rho\sigma_P\sigma_I(1 - \beta).$$

In the case where the index I is taken to be the market portfolio, Johnson and Tian note that β may be interpreted as the company's equity β from the capital asset pricing model (CAPM). Thus, the return associated with systematic risk is being filtered out of the option payoff.

However, suppose that the company desires to design indexed executive option compensation that depends on the company's capital structure and only filters out excess return based on the asset (unlevered) β, β_U, rather than the equity (levered) β, β_L as in the Johnson and Tian approach. For determining the risk-neutral value of such a contract, the pricing formula derived by Johnson and Tian would no longer be valid since it depends on their particular definition of the benchmark stock. Their approach could be applied, but it would require using the new conditions to derive another benchmark stock of the form that can be used in the standard exchange option formula.

Alternatively, if we define $\xi = (1 - \tau)D/E$ where τ and D/E are respectively the company's marginal tax rate and debt-equity ratio, then we can write

$$H_t = E[P_t | I_t,\, \alpha = 0] = P_0 \left(\frac{I_t}{I_0}\right)^{\beta_L} e^{\eta t} = \left[P_0^{1/(1+\xi)} \left(\frac{I_t}{I_0}\right)^{\beta_U} e^{(\eta/(1+\xi))t}\right]^{1+\xi},$$

and define a benchmark stock

$$H_t^{(1/(1+\xi))} = P_0^{1/(1+\xi)} \left(\frac{I_t}{I_0}\right)^{\beta_U} e^{(\eta/(1+\xi))t}$$

that is based on β_U. Then the new form of the contract can be priced as a power exchange option with $\lambda_1 = 1$, $S_1 = P$, $\alpha_1 = 1$, $\lambda_2 = P_0^{1+\xi}$, $S_2 = H$, $\alpha_2 = 1/(1 + \xi)$.

The foregoing example is intended to illustrate the relevance of power exchange options. Yet power exchange options are even more flexible than this example suggests since they allow nonlinear dependence on the index as well as the company's stock price.

5 Conclusion and Directions for Future Work

In this paper we present pricing results for an option to exchange the value of one asset raised to a power for the value of another asset raised to a power. Such power exchange options generalize results for both the Fischer-Margrabe-type option to exchange one asset for another and power options. After deriving closed-form results for the price of a European power exchange option, we used these results to price options paying the best or worst of powers of two assets. We then turned to the problem of optimality of early exercise for the American power exchange option and were able to generalize the results of Margrabe (1978) by establishing sufficient conditions for the suboptimality of early exercise. Along the lines of the analysis of the optimal exercise problem for the standard American put in Jacka (1991), it is possible to show that early exercise for the power exchange option is optimal for certain parameterizations. These results will be the subject of a forthcoming paper.

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